Lecture 12: SOS Lower Bounds for Planted Clique Part I

Lecture Outline

- Part I: Planted Clique and the Meka-Wigderson Moments
- Part II: MPW Analysis Preprocessing
- Part III: MPW Analysis with Graph Matrices
- Part IV: The Pessimist Strikes Back

Part I: Planted Clique and the Meka-Wigderson Moments

Review: Planted Clique

- Recall the planted clique problem: Given a random graph G where a clique of size k has been planted, can we find this planted clique?
- Variant we'll analyze: Can we use SOS to prove that a random $G\left(n, \frac{1}{2}\right)$ graph has no clique of size k where $k \gg 2logn$ (the expected size of the largest clique in a random graph)?

Review: Planted Clique Equations

- Variable x_i for each vertex i in G.
- Want $x_i = 1$ if i is in the clique.
- Want $x_i = 0$ if i is not in the clique.
- Equations:

$x_i^2 = x_i$ for all i.

$x_i x_j = 0$ if $(i, j) \notin E(G)$

$\sum_i x_i \geq k$

First SOS Lower Bound

- Theorem [MPW15]: $\exists C > 0$ such that whenever $k \leq C^d \left(\frac{n}{(logn)^2}\right)^{\frac{1}{d}}$, with high probability degree d SOS cannot prove the k-clique equations are infeasible.

Review: SOS Lower Bound Strategy

- To prove an SOS lower bound:
  1. Come up with pseudo-expectation values $\tilde{E}$ which obey the required linear equations
  2. Show that the moment matrix M is PSD

MW Moments

- Idea: Give each d-clique the same weight
- Define $x_I = \prod_{i \in I} x_i$
- Define $N_d(I)$ to be the number of d-cliques containing I.

- MW moments: take $\tilde{E}[x_I] = \frac{\binom{k}{|I|}}{\binom{d}{|I|}} \cdot \frac{N_d(I)}{N_d(\emptyset)}$

Checking $\sum_i x_i = k$

- MW moments: take $\tilde{E}[x_I] = \frac{\binom{k}{|I|}}{\binom{d}{|I|}} \cdot \frac{N_d(I)}{N_d(\emptyset)}$
- MW moments obey the equation $\sum_i x_i = k$
- Proof: $\sum_{i \notin I} N_d(I \cup i) = (d - |I|)N_d(I)$ as each d-clique containing I contains $d - |I|$ of the $i \notin I$
- $\frac{\binom{k}{|I|+1}}{\binom{d}{|I|+1}} = \frac{k-|I|}{d-|I|} \cdot \frac{\binom{k}{|I|}}{\binom{d}{|I|}}$
- $\sum_i \tilde{E}[x_{I \cup i}] = |I|\tilde{E}[x_I] + (k - |I|)\tilde{E}[x_I] = k\tilde{E}[x_I]$

Part II: MPW Analysis Preprocessing

Analysis Outline

- For the MPW analysis, we do the following:
  1. Preprocess the moment matrix M to make it easier to analyze. More specifically, we find a matrix M' which is easier to analyze such that if $\lambda_{\min}(M') \geq \frac{k^{\frac{d}{2}}}{4n^{\frac{d}{2}}}$ then $M \succcurlyeq 0$ with high probability
  2. Decompose $M' = E[M'] + R$ and show that $E[M'] \succcurlyeq \frac{k^{\frac{d}{2}}}{2n^{\frac{d}{2}}} Id$ and w.h.p., $\|R\| \leq \frac{k^{\frac{d}{2}}}{4n^{\frac{d}{2}}}$

Restriction to Multilinear, Degree $\frac{d}{2}$

- Preprocessing Step #1: As we've seen from the 3XOR and knapsack lower bounds, since we have the constraints that $x_i^2 = x_i$ for all i and $\sum_i x_i = k$, it is sufficient to consider the submatrix of M with multilinear, degree $\frac{d}{2}$ indices

Approximating $\tilde{E}[x_I]$

- Preprocessing Step #2: Approximate $\tilde{E}[x_I]$
- Intuition: One view of $\tilde{E}[x_I]$ is that $\tilde{E}[x_I]$ is the expected value of x_I given what we can compute.
- Remark: This is connected to pseudo-calibration/moment matching which we'll see next lecture.

Approximating $\tilde{E}[x_I]$ Continued

- A priori, if we choose a clique of size k at random, $|I|$ is part of the clique with probability $\frac{\binom{k}{|I|}}{\binom{n}{|I|}} \approx \frac{k^{|I|}}{n^{|I|}}$
- If I is not a clique, $\tilde{E}[x_I] = 0$. If I is a clique, I is $2^{\binom{|I|}{2}}$ times more likely to be part of the clique. Thus, $\tilde{E}[x_I] \approx 2^{\binom{|I|}{2}} \frac{k^{|I|}}{n^{|I|}}$ if I is a clique and is 0 otherwise.
- See appendix for calculations confirming this.

Approximation Error

- Let M_{approx} be the matrix where $\left(M_{approx}\right)_{IJ} = 2^{\binom{|I \cup J|}{2}} \frac{k^{|I \cup J|}}{n^{|I \cup J|}}$ if $I \cup J$ is a clique and $\left(M_{approx}\right)_{IJ} = 0$ otherwise.
- Can show that the difference $\Delta = \mathrm{M} - \mathrm{M}_{approx}$ is small (see [MPW15] for details).

The matrix M'

- Preprocessing Step #3: Fill in zero rows and columns of M_{approx}
- If I or J is not a clique then $(M_{approx})_{IJ} = 0$.
- These zero rows and columns make M_{approx} harder to analyze.
- Definition: Take M' to be the matrix such that $M'_{IJ} = 2^{\binom{|I \cup J|}{2}} \frac{k^{|I \cup J|}}{n^{|I \cup J|}}$ if all edges are present between $I \setminus J$ and $J \setminus I$ and $M'_{IJ} = 0$ otherwise

$$M' \succcurlyeq 0 \Rightarrow M_{approx} \succcurlyeq 0$$

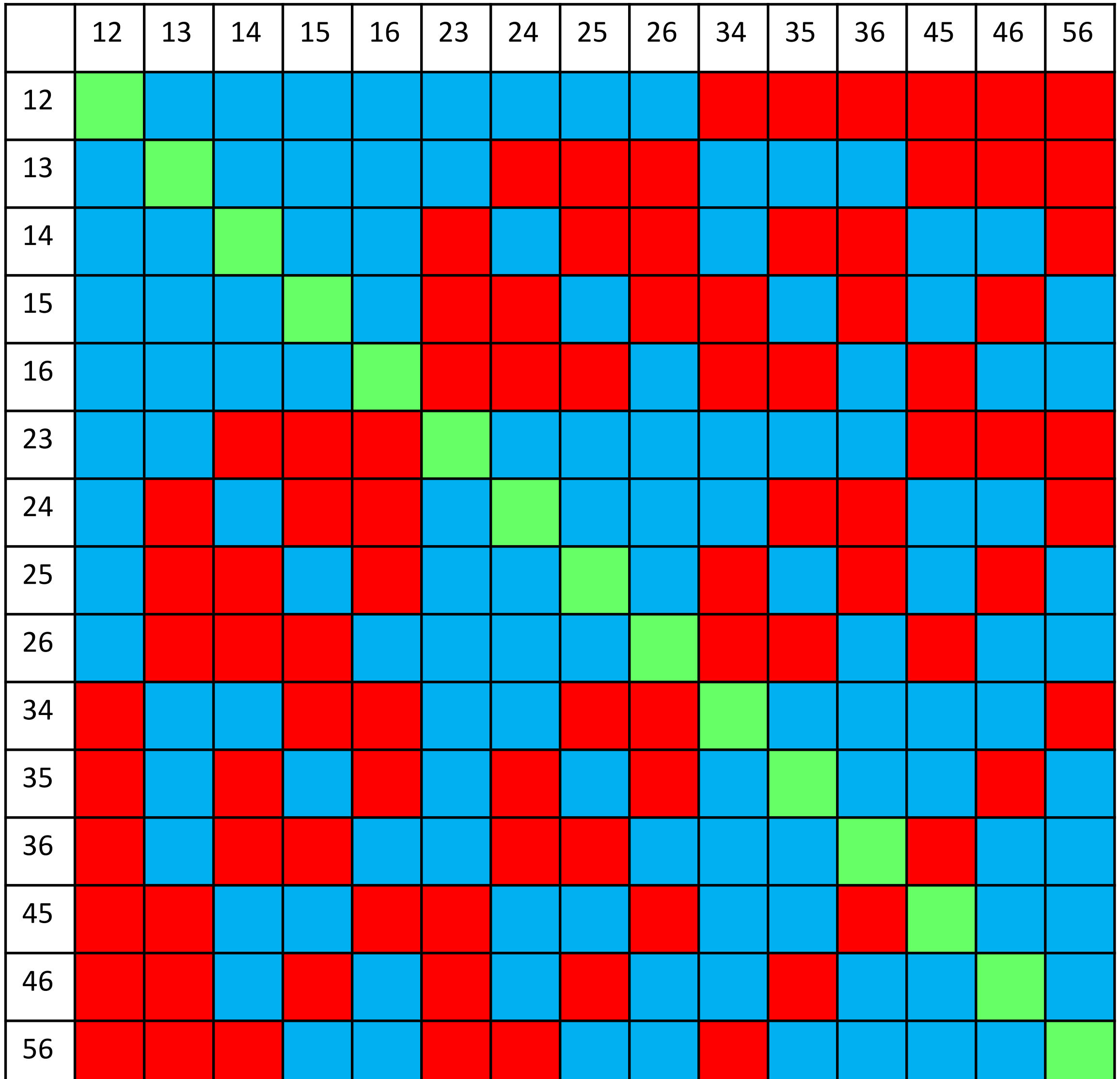
- Can view M_{approx} as a submatrix of M'.
- This immediately implies that if $M' \succcurlyeq 0$ then $M_{approx} \succcurlyeq 0$
- Because of the error matrix $\Delta = \mathrm{M} - \mathrm{M}_{approx}$ we need the stronger statement that with high probability, $\lambda_{\min}(M')$ is significantly bigger than 0.

Summary

- We want to show that w.h.p. $M' \succcurlyeq \frac{k^{\frac{d}{2}}}{4n^{\frac{d}{2}}}$ where

  M' is the matrix such that $M'_{IJ} = 2^{\binom{|I \cup J|}{2}} \frac{k^{|I \cup J|}}{n^{|I \cup J|}}$ if all edges are present between $I \setminus J$ and $J \setminus I$ and $M'_{IJ} = 0$ otherwise

M' Picture for d = 4



$M'_{\{i,j\}\{i,j\}} = \frac{2k^2}{n^2}$

$M'_{\{i,j\}\{i,k\}} = \frac{8k^3}{n^3}$ if j $\sim k$ and 0 otherwise

$M'_{\{i,j\}\{k,l\}} = \frac{64k^4}{n^4}$ if $i \sim j, i \sim k, j \sim k, j \sim l$ and is 0 otherwise

Part III: MPW Analysis with Graph Matrices

Recall Definition of R_H

- Definition: Definition: If $V(H) = U \cup V$ then define $R_H(A,B) = \chi_{\sigma(E(H))}$ where $\sigma\text{: V(H)} \rightarrow V(G)$ is the injective map satisfying $\sigma(U) = A$, $\sigma(V) = B$ and preserving the ordering of U, V.
- Last lecture: Did not require A, B to be in ascending order.
- This lecture: Will require A, B to be in ascending order.
- Note: This only reduces our norms, so the probabilistic norm bounds still hold.

Review: Rough Norm Bound

- Theorem [MP16]: If H has no isolated vertices then with high probability, $\|R_H\|$ is $\tilde{O}\left(n^{(|V(H)|-s_H)/2}\right)$ where s_H is the minimal size of a vertex separator between U and V (S is a vertex separator of U and V if every path from U to V intersects S)
- Note: The $\tilde{O}$ contains polylog factors and constants related to the size of H.

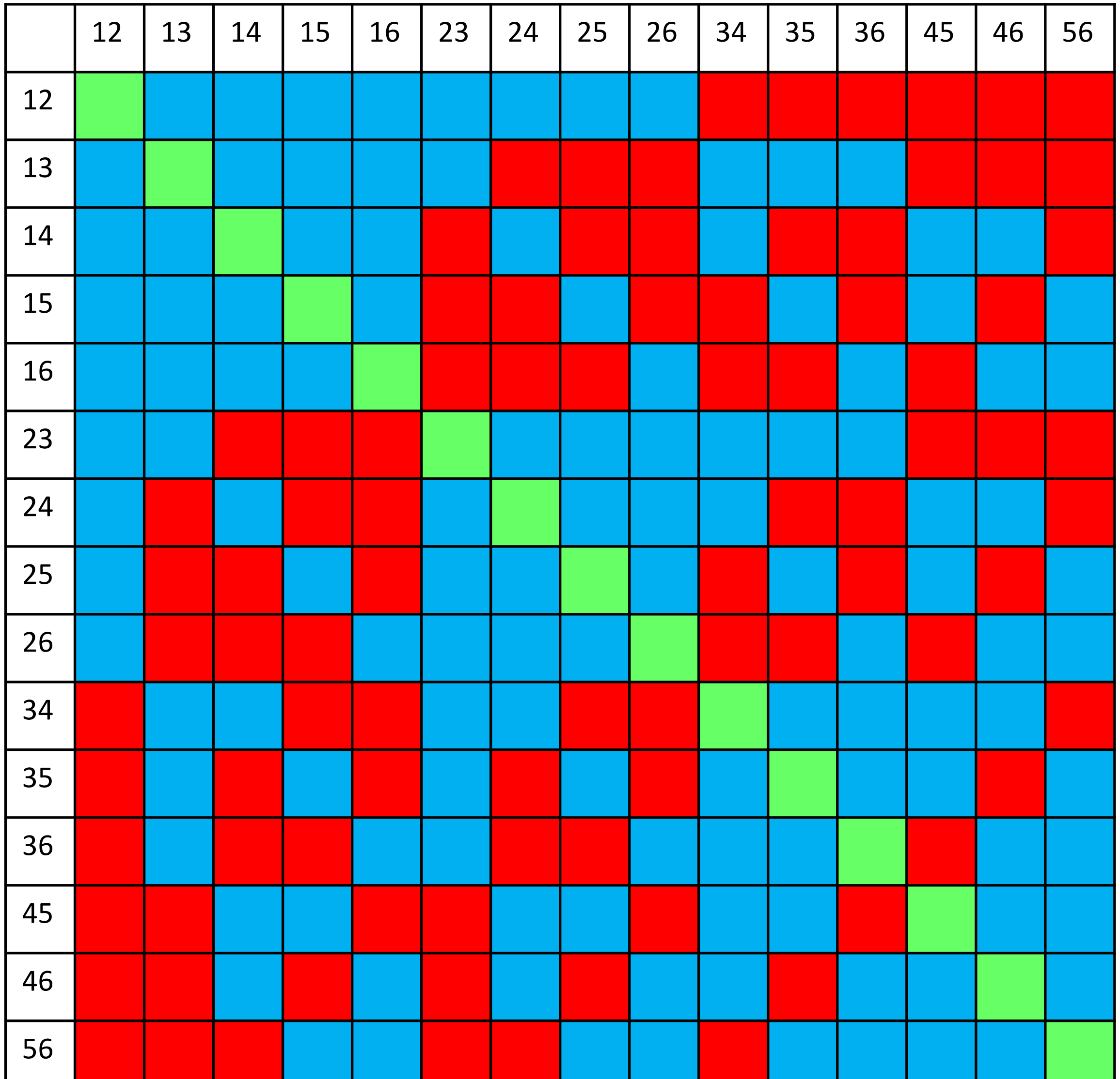
Decomposition of M_{approx} and M'

- Claim: $M_{approx} = \sum_H \frac{k^{|U \cup V|}}{n^{|U \cup V|}} R_H$ where we sum over H which have no middle vertices.
- Claim: $M' = \sum_H 2^{\binom{|U|}{2} + \binom{|V|}{2} - \binom{|U \cap V|}{2}} \frac{k^{|U \cup V|}}{n^{|U \cup V|}} R_H$ where we sum over H which have no middle vertices and which have no edges within U or within V.
- Idea: Each of the $2^{\binom{|U|}{2} + \binom{|V|}{2} - \binom{|U \cap V|}{2}}$ edges within U or V are given for free.

Entries of $\mathrm{E}[M']$

- $M' = \sum_H 2^{\binom{|U|}{2}+\binom{|V|}{2}-\binom{|U\cap V|}{2}} \frac{k^{|U\cup V|}}{n^{|U\cup V|}} R_H$ where we sum over H which have no middle vertices and which have no edges within U or within V.
- Claim: $\mathrm{E}[M']_{IJ} = 2^{\binom{|I|}{2}+\binom{|J|}{2}-\binom{|I\cap J|}{2}} \frac{k^{|I\cup J|}}{n^{|I\cup J|}}$
- Idea: For any H which has an edge, $E[R_H] = 0$. Otherwise, $E[R_H] = R_H$

$E[M']$ Picture for d = 4



$E[M']_{\{i,j\}\{i,j\}} = \frac{2k^2}{n^2}$

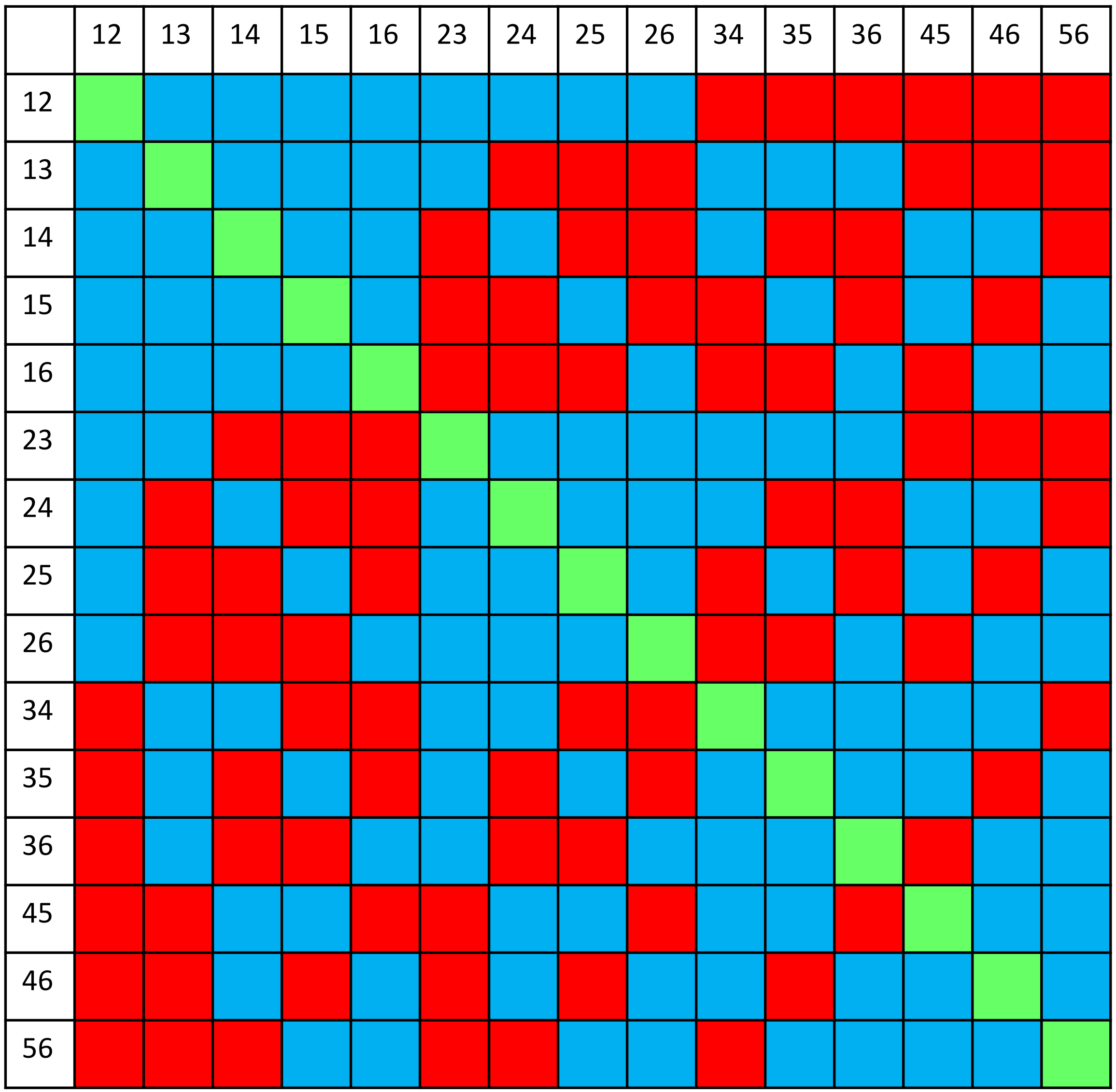
$E[M']_{\{i,j\}\{i,k\}} = \frac{4k^3}{n^3}$

$E[M']_{\{i,j\}\{k,l\}} = \frac{4k^4}{n^4}$

Analysis of $E[M']$

- $E[M']$ belongs to the Johnson Scheme of matrices A whose entries A_{IJ} only depend on $|I \cap J|$ (See Lecture 9 on SOS Lower Bounds for Knapsack)
- Can decompose $E[M']$ as a sum of PSD matrices, one of which is the identity matrix which has coefficient $\geq \frac{k^{\frac{d}{2}}}{2n^{\frac{d}{2}}} Id$.

One Piece of $M' - E[M']$ $(d = 4)$



0

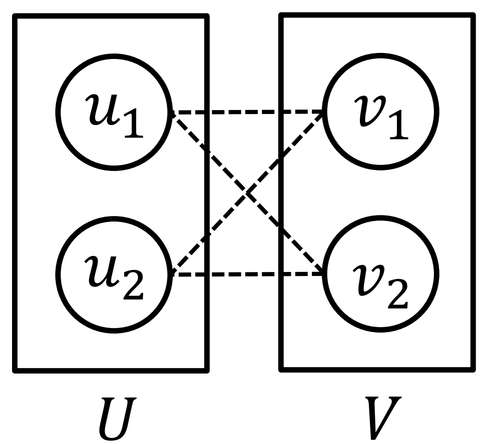
0

$\frac{60k^4}{n^4}$ if all edges between I and J are present.

$-\frac{4k^4}{n^4}$ otherwise

Piece of $M' - E[M']$ Decomposition

- This piece has coefficient $\frac{4k^4}{n^4}$ in R_H for all H which have the following form (and 0 for all other R_H):



Where $E(H)$ is non-empty and is a subset of the dashed lines

Piece of $M' - E[M']$ Analysis

- All H here have minimum separator size s_H at least 1.
- This gives a norm bound of $\tilde{O}\left(\frac{k^4}{n^4} \cdot n^{\frac{4-1}{2}}\right) = \tilde{O}\left(\frac{k^2}{\sqrt{n}} \cdot \frac{k^2}{n^2}\right)$
- This is much less than $\frac{k^2}{4n^2}$ when $k \ll n^{\frac{1}{4}}$.

General Analysis of $R = M' - E[M']$

- Define $R = M' - E[M']$
- Claim: $R = \sum_H 2^{\binom{|U|}{2}+\binom{|V|}{2}-\binom{|U\cap V|}{2}} \frac{k^{|U\cup V|}}{n^{|U\cup V|}} R_H$ where we sum over H which have no middle vertices, which have no edges within U or within V, and which have at least one edge.

General Analysis of $R = M' - E[M']$

- $R = \sum_H 2^{\binom{|U|}{2}+\binom{|V|}{2}-\binom{|U \cap V|}{2}} \frac{k^{|U \cup V|}}{n^{|U \cup V|}} R_H$ where we sum over H which have no middle vertices, which have no edges within U or within V, and which have at least one edge
- Norm bound: For any such R_H, w.h.p. $||R_H||$ is $\tilde{O}(n^{\frac{|U \cup V| - |U \cap V| - 1}{2}})$ as the minimal separator size s_H between U and V is at least $|U \cap V| + 1$
- Corollary: w.h.p. $\frac{k^{|U \cup V|}}{n^{|U \cup V|}} R_H$ is $\tilde{O}\left(\frac{k^{|U \cup V|}}{\sqrt{n}^{|U \cup V| + |U \cap V| + 1}}\right)$

General Analysis of $R = M' - E[M']$

- R is a sum of terms which w.h.p. have norm $\tilde{O}\left(\frac{k^{|U \cup V|}}{\sqrt{n}^{|U \cup V| + |U \cap V| + 1}}\right)$
- $|U \cup V| \leq d$ and $|U \cup V| + |U \cap V| = d$, so w.h.p. $\|R\|$ is $\tilde{O}\left(\frac{k^{\frac{d}{2}}}{n^{\frac{d}{2}}} \cdot \frac{k^{\frac{d}{2}}}{\sqrt{n}}\right)$. This is much less than $\frac{k^{\frac{d}{2}}}{4n^{\frac{d}{2}}}$ as long as $k \ll n^{\frac{1}{d}}$

Part IV: The Pessimist Strikes Back

Limitations of MW moments

- Can we prove a stronger lower bound with the MW moments?
- With a more careful analysis, a slightly stronger lower bound can be shown. For $d = 4$, [DM15] proved an $\widetilde{\Omega}(n^{\frac{1}{3}})$ lower bound. [HKPRS16] generalized this to $\widetilde{\Omega}(n^{\frac{2}{d+2}})$
- By an argument of Jonathan Kelner, this is tight!

Pessimist's Query

- Kelner's argument: Pessimist can query the following polynomial:
- Take $p = Cx_i - \sum_{J:|J|=\frac{d}{2}, i \notin J} (-1)^{|J \setminus N(I)|} x_J$ where $N(I)$ is the set of neighbors of I
- What is $\tilde{E}\left[p^2\right]$?
- Key idea: Cross terms will all be negative, but there will be cancellation in the square terms.

Pessimist's Query Analysis

- $p = Cx_i - \sum_{J:|J|=\frac{d}{2}, i \notin J} (-1)^{|J \setminus N(i)|} x_J$ where $N(i)$ is the set of neighbors of I

$p^2 = C^2 x_i - 2C \sum_{J: J \cup \{i\} \text{ } is \text{ } a \text{ } clique} x_{J \cup \{i\}} + \sum_{J,J'} (-1)^{|(J \Delta J') \setminus N(I)|} x_{J \cup J'}$

- We expect $\tilde{E}[C^2 x_i]$ to be $\Theta\left(\frac{C^2 k}{n}\right)$
- We expect $\tilde{E}\left[2C \sum_{J: J \cup \{i\} \text{ } is \text{ } a \text{ } clique} x_{J \cup \{i\}}\right]$ to be $\Theta\left(\frac{Ck^{(d/2)+1}}{n}\right)$

Pessimist's Query Analysis Continued

- $p^2 = C^2 x_i - 2C \sum_{J: J \cup \{i\} \text{ } is\ a\ clique} x_{J \cup \{i\}} + \sum_{J,J'} (-1)^{|(J \Delta J') \setminus N(I)|} x_{J \cup J'}$
- All terms of $\sum_{J,J'} \tilde{E}\left[(-1)^{|(J \Delta J') \setminus N(I)|} x_{J \cup J'}\right]$ have expected value ≈ 0 except for the ones where $J' = J$.
- These terms contribute $\Theta(k^{d/2})$ and it turns out that w.h.p. these terms are dominant

Pessimist's Query Analysis Continued

- We expect $\tilde{E}[p^2]$ to be $\Theta\left(\frac{C^2 k}{n}\right) - \Theta\left(\frac{C k^{\left(\frac{d}{2}\right)+1}}{n}\right) + \Theta(k^{d/2})$
- Taking $C = k^{\frac{d}{4}-\frac{1}{2}}\sqrt{n}$, this is

$$\Theta(k^{d/2}) - \Theta\left(\frac{k^{\left(\frac{3d}{4}\right)+\frac{1}{2}}}{\sqrt{n}}\right) = k^{d/2}\Theta\left(1 - \frac{k^{\left(\frac{d+2}{4}\right)}}{\sqrt{n}}\right)$$

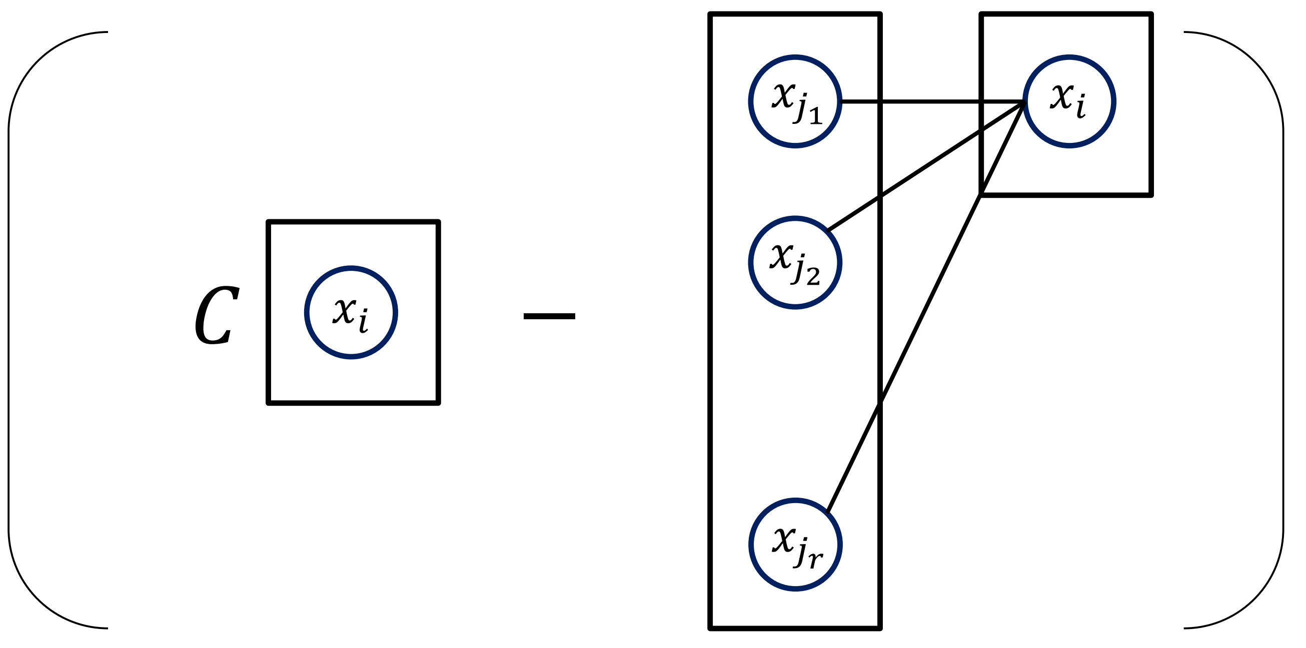
which is negative if $k \gg n^{\frac{2}{d+2}}$

Back to the Drawing Board

- Pessimist has disproven our (Optimist's) first attempt at bluffing, but perhaps we can come up with a better bluff.
- Let's see what went wrong.

Graphical Picture

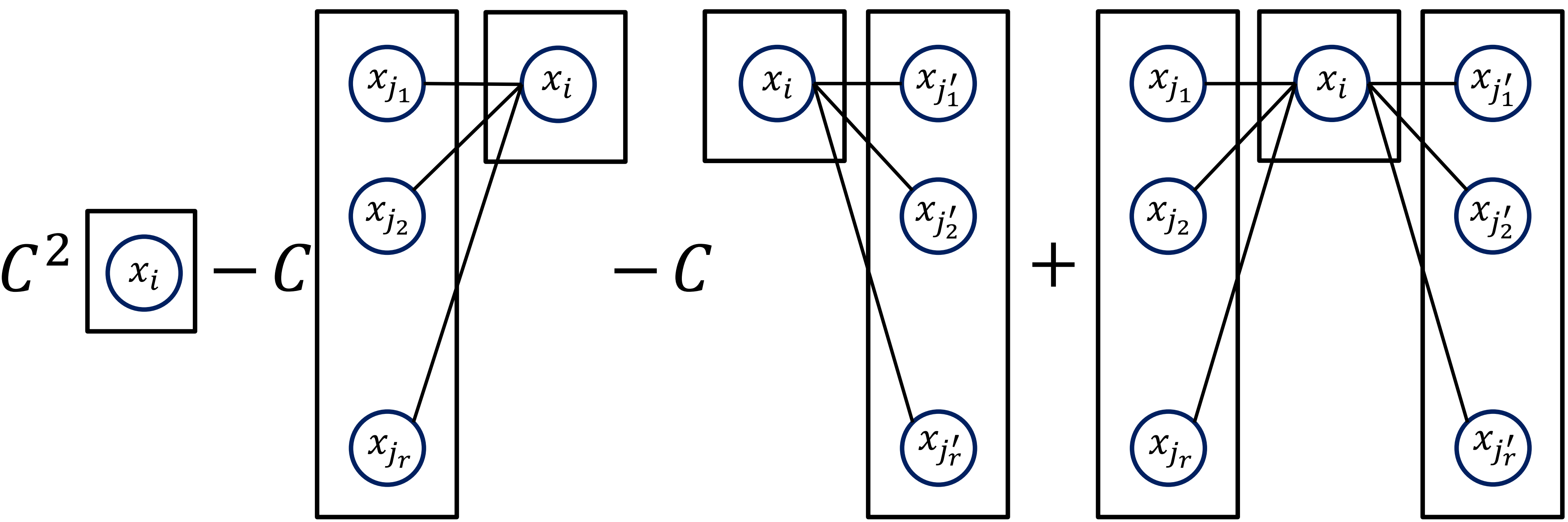
- Can represent the polynomial Pessimist is querying as follows:



times its transpose

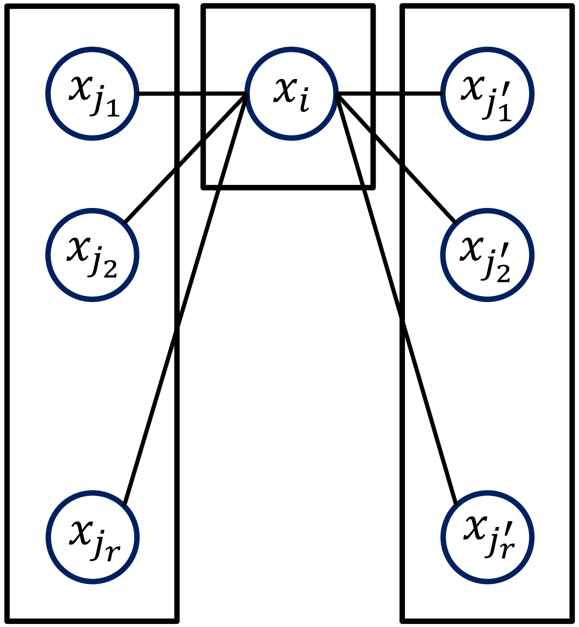
Graphical Picture

- Multiplying graph matrices is tricky (more on that next lecture!). Some terms that appear are:



Potential Fix

- What if we add an appropriate multiple of



to our moment matrix?

Potential Fix Analysis

- This fix does work for $d = 4$ [HKPRS16]
- However, it seems rather ad-hoc.
- Remark: It is related to giving more weight to cliques which have more common neighbors, but that’s not quite what it does…
- Can we find a more principled general fix? Yes, see next lecture!

References

- [BHK+16] B. Barak, S. B. Hopkins, J. A. Kelner, P. Kothari, A. Moitra, and A. Potechin, A nearly tight sum-of-squares lower bound for the planted clique problem, FOCS p.428–437, 2016.
- [DM15] Y. Deshpande and A. Montanari, Improved sum-of-squares lower bounds for hidden clique and hidden submatrix problems, COLT, JMLR Workshop and Conference Proceedings, vol.40, JMLR.org, p.523–562,2015.
- [HKPRS16] S. Hopkins, P. Kothari, A. Potechin, P. Raghavendra, T. Schramm. Tight Lower Bounds for Planted Clique in the Degree-4 SOS Program. SODA 2016
- [MP16] D. Medarametla, A. Potechin. Bounds on the Norms of Uniform Low Degree Graph Matrices. RANDOM 2016. https://arxiv.org/abs/1604.03423
- [MPW15] R. Meka, Aaron Potechin, and Avi Wigderson, Sum-of-squares lower bounds for planted clique. STOC p.87–96, 2015

Appendix

Approximating $\tilde{E}[x_I]$ Calculation

- $\tilde{E}[x_I] = \frac{\binom{k}{|I|}}{\binom{d}{|I|}} \cdot \frac{N_d(I)}{N_d(\emptyset)}$
- If I is a clique then $N_d(I) \approx 2^{\binom{|I|}{2}-\binom{d}{2}} \binom{n-|I|}{d-|I|}$
- As a special case, $N_d(\emptyset) \approx 2^{-\binom{d}{2}} \binom{n}{d}$
- If I is a clique then

$$\tilde{E}[x_I] \approx \frac{\binom{k}{|I|} 2^{\binom{|I|}{2}-\binom{d}{2}} \binom{n-|I|}{d-|I|}}{\binom{d}{|I|} 2^{-\binom{d}{2}} \binom{n}{d}} = 2^{\binom{|I|}{2}} \frac{\binom{k}{|I|}}{\binom{n}{|I|}} \approx 2^{\binom{|I|}{2}} \frac{k^{|I|}}{n^{|I|}}$$