Lecture 12: SOS Lower Bounds for Planted Clique Part I
Lecture Outline

• Part I: Planted Clique and the Meka-Wigderson Moments
• Part II: MPW Analysis Preprocessing
• Part III: MPW Analysis with Graph Matrices
• Part IV: The Pessimist Strikes Back
Part I: Planted Clique and the Meka-Wigderson Moments
Review: Planted Clique

• Recall the planted clique problem: Given a random graph $G$ where a clique of size $k$ has been planted, can we find this planted clique?

• Variant we’ll analyze: Can we use SOS to prove that a random $G \left( n, \frac{1}{2} \right)$ graph has no clique of size $k$ where $k \gg 2 \log n$ (the expected size of the largest clique in a random graph)?
Review: Planted Clique Equations

- Variable $x_i$ for each vertex $i$ in $G$.
- Want $x_i = 1$ if $i$ is in the clique.
- Want $x_i = 0$ if $i$ is not in the clique.
- Equations:
  
  $x_i^2 = x_i$ for all $i$.
  
  $x_i x_j = 0$ if $(i,j) \notin E(G)$.
  
  $\sum_i x_i \geq k$
First SOS Lower Bound

- Theorem [MPW15]: \( \exists C > 0 \) such that whenever
  \[
  k \leq C^d \left( \frac{n}{(\log n)^2} \right)^{\frac{1}{d}},
  \]
  with high probability degree \( d \) SOS cannot prove the \( k \)-clique equations are infeasible.
Review: SOS Lower Bound Strategy

• To prove an SOS lower bound:
  1. Come up with pseudo-expectation values $\tilde{E}$ which obey the required linear equations
  2. Show that the moment matrix $M$ is PSD
MW Moments

• Idea: Give each $d$-clique the same weight
• Define $x_I = \prod_{i \in I} x_i$
• Define $N_d(I)$ to be the number of $d$-cliques containing $I$.

• MW moments: take $\tilde{E}[x_I] = \frac{k}{{|I| \choose d}} \cdot \frac{N_d(I)}{N_d(\emptyset)}$
Checking $\sum_i x_i = k$

- MW moments: take $\tilde{E}[x_I] = \left(\frac{k}{|I|}\right) \cdot \frac{Nd(I)}{Nd(\emptyset)}$
- MW moments obey the equation $\sum_i x_i = k$
- Proof: $\sum_{i \notin I} Nd(I \cup i) = (d - |I|)Nd(I)$ as each $d$-clique containing $I$ contains $d - |I|$ of the $i \notin I$
- $\left(\frac{k}{|I|+1}\right) \left(\frac{d}{|I|+1}\right) = \frac{k-|I|}{d-|I|} \cdot \left(\frac{k}{|I|}\right)$
- $\sum_i \tilde{E}[x_{I\cup i}] = |I|\tilde{E}[x_I] + (k - |I|)\tilde{E}[x_I] = k\tilde{E}[x_I]$
Part II: MPW Analysis Preprocessing
Analysis Outline

• For the MPW analysis, we do the following:
  
  1. Preprocess the moment matrix $M$ to make it easier to analyze. More specifically, we find a matrix $M'$ which is easier to analyze such that if

\[
\lambda_{\min}(M') \geq \frac{k^2}{d} \frac{d}{4n^2}
\]

then $M \succeq 0$ with high probability

  2. Decompose $M' = E[M'] + R$ and show that

\[
E[M'] \succeq \frac{k^2}{d} \frac{d}{2n^2} Id \quad \text{and w.h.p., } \|R\| \leq \frac{k^2}{d} \frac{d}{4n^2}
\]
Preprocessing Step #1: As we’ve seen from the 3XOR and knapsack lower bounds, since we have the constraints that $x_i^2 = x_i$ for all $i$ and $\sum_i x_i = k$, it is sufficient to consider the submatrix of $M$ with multilinear, degree $\frac{d}{2}$ indices.
Approximating $\tilde{E}[x_I]$

- Preprocessing Step #2: Approximate $\tilde{E}[x_I]$
- Intuition: One view of $\tilde{E}[x_I]$ is that $\tilde{E}[x_I]$ is the expected value of $x_I$ given what we can compute.
- Remark: This is connected to pseudo-calibration/moment matching which we’ll see next lecture.
Approximating $\tilde{E}[x_I]$ Continued

- A priori, if we choose a clique of size $k$ at random, $|I|$ is part of the clique with probability $\frac{\binom{k}{|I|}}{\binom{n}{|I|}} \approx \frac{k|I|}{n|I|}$

- If $I$ is not a clique, $\tilde{E}[x_I] = 0$. If $I$ is a clique, $I$ is $2^{\frac{|I|}{2}}$ times more likely to be part of the clique. Thus, $\tilde{E}[x_I] \approx 2^{\frac{|I|}{2}} \frac{k|I|}{n|I|}$ if $I$ is a clique and is 0 otherwise.

- See appendix for calculations confirming this.
Approximation Error

• Let $M_{\text{approx}}$ be the matrix where
  
  $$(M_{\text{approx}})_{IJ} = 2\binom{|I \cup J|}{2} \cdot \frac{k|I \cup J|}{n|I \cup J|}$$  
  if $I \cup J$ is a clique and $(M_{\text{approx}})_{IJ} = 0$ otherwise.

• Can show that the difference $\Delta = M - M_{\text{approx}}$ is small (see [MPW15] for details).
The matrix $M'$

- Preprocessing Step #3: Fill in zero rows and columns of $M_{\text{approx}}$
- If $I$ or $J$ is not a clique then $(M_{\text{approx}})_{IJ} = 0$.
- These zero rows and columns make $M_{\text{approx}}$ harder to analyze.
- Definition: Take $M'$ to be the matrix such that
  \[ M'_{IJ} = 2^{(|I \cup J|)} \frac{k^{|I \cup J|}}{n^{|I \cup J|}} \text{ if all edges are present between } I \setminus J \text{ and } J \setminus I \text{ and } M'_{IJ} = 0 \text{ otherwise} \]
\[ M' \succeq 0 \Rightarrow M_{approx} \succeq 0 \]

- Can view \( M_{approx} \) as a submatrix of \( M' \).
- This immediately implies that if \( M' \succeq 0 \) then \( M_{approx} \succeq 0 \).
- Because of the error matrix \( \Delta = M - M_{approx} \) we need the stronger statement that with high probability, \( \lambda_{\text{min}}(M') \) is significantly bigger than 0.
Summary

• We want to show that w.h.p. $M' \geq \frac{d}{d} \frac{k \frac{|I \cup J|}{2}}{4n^2} \frac{|I \cup J|}{n |I \cup J|}$ where $M'$ is the matrix such that $M'_{IJ} = 2 \binom{|I \cup J|}{2} \frac{k |I \cup J|}{n |I \cup J|}$ if all edges are present between $I \setminus J$ and $J \setminus I$ and $M'_{IJ} = 0$ otherwise.
$M'$ Picture for $d = 4$

\[
M'_{i,j}\{i,j\} = \frac{2k^2}{n^2}
\]

\[
M'_{i,j}\{i,k\} = \frac{8k^3}{n^3}
\]
if $j \sim k$ and 0 otherwise

\[
M'_{i,j}\{k,l\} = \frac{64k^4}{n^4}
\]
if $i \sim j, i \sim k, j \sim k, j \sim l$ and is 0 otherwise
Part III: MPW Analysis with Graph Matrices
Recall Definition of $R_H$

- Definition: Definition: If $V(H) = U \cup V$ then define $R_H(A, B) = \chi_{\sigma(E(H))}$ where $\sigma: V(H) \to V(G)$ is the injective map satisfying $\sigma(U) = A$, $\sigma(V) = B$ and preserving the ordering of $U, V$.
- Last lecture: Did not require $A, B$ to be in ascending order.
- This lecture: Will require $A, B$ to be in ascending order.
- Note: This only reduces our norms, so the probabilistic norm bounds still hold.
Review: Rough Norm Bound

• Theorem [MP16]: If $H$ has no isolated vertices then with high probability, $\|R_H\|$ is $\tilde{O}(n(|V(H)|−s_H)/2)$ where $s_H$ is the minimal size of a vertex separator between $U$ and $V$ (S is a vertex separator of U and V if every path from U to V intersects S)

• Note: The $\tilde{O}$ contains polylog factors and constants related to the size of $H$. 
Decomposition of $M_{approx}$ and $M'$

- Claim: $M_{approx} = \sum_H \frac{k|U\cup V|}{n|U\cup V|} R_H$ where we sum over $H$ which have no middle vertices.

- Claim: $M' = \sum_H 2^{\binom{|U|}{2} + \binom{|V|}{2} - \binom{|U\cap V|}{2}} \frac{k|U\cup V|}{n|U\cup V|} R_H$ where we sum over $H$ which have no middle vertices and which have no edges within $U$ or within $V$.

- Idea: Each of the $2^{\binom{|U|}{2} + \binom{|V|}{2} - \binom{|U\cap V|}{2}}$ edges within $U$ or $V$ are given for free.
Entries of $E[M']$

- $M' = \sum_H 2^\left(\frac{|U|}{2}\right) + \left(\frac{|V|}{2}\right) - \left(\frac{|U \cap V|}{2}\right) \frac{k|U \cup V|}{n|U \cup V|} R_H$ where we sum over $H$ which have no middle vertices and which have no edges within $U$ or within $V$.

- Claim: $E[M']_{IJ} = 2^\left(\frac{|I|}{2}\right) + \left(\frac{|J|}{2}\right) - \left(\frac{|I \cap J|}{2}\right) \frac{k|I \cup J|}{n|I \cup J|}$

- Idea: For any $H$ which has an edge, $E[R_H] = 0$. Otherwise, $E[R_H] = R_H$
$E[M']$ Picture for $d = 4$

\[ E[M']_{\{i,j\}\{i,j\}} = \frac{2k^2}{n^2} \]

\[ E[M']_{\{i,j\}\{i,k\}} = \frac{4k^3}{n^3} \]

\[ E[M']_{\{i,j\}\{k,l\}} = \frac{4k^4}{n^4} \]
Analysis of $E[M']$

- $E[M']$ belongs to the Johnson Scheme of matrices $A$ whose entries $A_{IJ}$ only depend on $|I \cap J|$ (See Lecture 9 on SOS Lower Bounds for Knapsack)

- Can decompose $E[M']$ as a sum of PSD matrices, one of which is the identity matrix which has coefficient $\geq \frac{\kappa^2}{d} \frac{d}{2n^2} Id$. 
One Piece of $M' - E[M']$ ($d = 4$)

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- $\frac{60k^4}{n^4}$ if all edges between $I$ and $J$ are present.
- $\frac{4k^4}{n^4}$ otherwise
Piece of $\mathcal{M}' - E[\mathcal{M}']$ Decomposition

• This piece has coefficient $\frac{4k^4}{n^4}$ in $R_H$ for all $H$ which have the following form (and 0 for all other $R_H$):

Where $E(H)$ is non-empty and is a subset of the dashed lines.
Piece of $M' - E[M']$ Analysis

• All $H$ here have minimum separator size $s_H$ at least 1.

• This gives a norm bound of $	ilde{O} \left( \frac{k^4}{n^4} \cdot n^{\frac{4-1}{2}} \right) = \tilde{O} \left( \frac{k^2}{\sqrt{n}} \cdot \frac{k^2}{n^2} \right)$

• This is much less than $\frac{k^2}{4n^2}$ when $k \ll n^{\frac{1}{4}}$. 
General Analysis of $R = M' - E[M']$

- Define $R = M' - E[M']$

- Claim: $R = \sum_H 2^{(\frac{|U|}{2}) + (\frac{|V|}{2}) - (\frac{|U \cap V|}{2})} \frac{k|U \cup V|}{n|U \cup V|} R_H$

  where we sum over $H$ which have no middle vertices, which have no edges within $U$ or within $V$, and which have at least one edge.
General Analysis of $R = M' - E[M']$

- $R = \sum_H 2^{|U|/2} + |V| - (|U \cap V|) \frac{k|U \cup V|}{n|U \cup V|} R_H$ where we sum over $H$ which have no middle vertices, which have no edges within $U$ or within $V$, and which have at least one edge.

- Norm bound: For any such $R_H$, w.h.p. $\|R_H\|$ is $\tilde{O}(n \frac{|U \cup V| - |U \cap V| - 1}{2})$ as the minimal separator size $s_H$ between $U$ and $V$ is at least $|U \cap V| + 1$.

- Corollary: w.h.p. $\frac{k|U \cup V|}{n|U \cup V|} R_H$ is $\tilde{O}\left(\frac{k|U \cup V|}{\sqrt{n}|U \cup V| + |U \cap V| + 1}\right)$.
General Analysis of $R = M' - E[M']$

- $R$ is a sum of terms which w.h.p. have norm
  \[ \tilde{O}\left(\frac{k|U \cup V|}{\sqrt{n}|U \cup V| + |U \cap V| + 1}\right) \]

- $|U \cup V| \leq d$ and $|U \cup V| + |U \cap V| = d$, so
  w.h.p. $\|R\|$ is $\tilde{O}\left(\frac{d}{k^2} \cdot \frac{d}{\sqrt{n}}\right)$. This is much less than
  \[ \frac{d}{k^2d} \frac{1}{4n^2} \]
as long as $k \ll \frac{1}{n^d}$.
Part IV: The Pessimist Strikes Back
Limitations of MW moments

• Can we prove a stronger lower bound with the MW moments?
• With a more careful analysis, a slightly stronger lower bound can be shown. For $d = 4$, [DM15] proved an $\tilde{\Omega}(n^{1/3})$ lower bound. [HKPRS16] generalized this to $\tilde{\Omega}(n^{d+2})$
• By an argument of Jonathan Kelner, this is tight!
Pessimist’s Query

• Kelner’s argument: Pessimist can query the following polynomial:

Take \( p = C x_i - \sum_{J:|J|=\frac{d}{2},i \notin J} (-1)^{|J\setminus N(I)|} x_J \) where \( N(I) \) is the set of neighbors of \( I \)

• What is \( \tilde{E}[p^2] \)?

• Key idea: Cross terms will all be negative, but there will be cancellation in the square terms.
Pessimist’s Query Analysis

• $p = Cx_i - \sum_{J: |J| = \frac{d}{2}, i \notin J} (-1)^{|J\setminus N(i)|} x_J$ where $N(i)$ is the set of neighbors of $I$

• $p^2 = C^2x_i - 2C \sum_{J: J \cup \{i\} \text{ is a clique}} x_{J \cup \{i\}} + \sum_{J, J'} (-1)^{|(J \Delta J') \setminus N(I)|} x_{J \cup J'}$

• We expect $\tilde{E}[C^2x_i]$ to be $\Theta\left(\frac{C^2k}{n}\right)$

• We expect $\tilde{E}\left[2C \sum_{J: J \cup \{i\} \text{ is a clique}} x_{J \cup \{i\}}\right]$ to be $\Theta\left(\frac{Ck^{(d/2)+1}}{n}\right)$
Pessimist’s Query Analysis Continued

- \( p^2 = C^2 x_i - 2C \sum_{J: J \cup \{i\} \text{ is a clique}} x_{J \cup \{i\}} + \sum_{J, J'} (-1)^{|J \Delta J'\rangle \setminus N(I)} x_{J \cup J'} \)
- All terms of \( \sum_{J, J'} \tilde{E} \left[ (-1)^{|J \Delta J'\rangle \setminus N(I)} x_{J \cup J'} \right] \) have expected value \( \approx 0 \) except for the ones where \( J' = J \).
- These terms contribute \( \Theta(k^{d/2}) \) and it turns out that w.h.p. these terms are dominant.
Pessimist’s Query Analysis Continued

- We expect $\tilde{E}[p^2]$ to be $\Theta \left( \frac{C^2 k}{n} \right) - \Theta \left( \frac{C k (\frac{d}{2}) + 1}{n} \right) + \Theta(k^{d/2})$

- Taking $C = k^4 \frac{1}{2} \sqrt{n}$, this is

$$\Theta(k^{d/2}) - \Theta \left( \frac{k \left( \frac{3d}{4} \right) + \frac{1}{2}}{\sqrt{n}} \right) = k^{d/2} \Theta \left( 1 - \frac{k \left( \frac{d+2}{4} \right) }{\sqrt{n}} \right)^2$$

which is negative if $k \gg n^{d+2}$
Back to the Drawing Board

• Pessimist has disproven our (Optimist’s) first attempt at bluffing, but perhaps we can come up with a better bluff.

• Let’s see what went wrong.
Graphical Picture

- Can represent the polynomial Pessimist is querying as follows:

$$C \times C^{\top}$$
Multiplying graph matrices is tricky (more on that next lecture!). Some terms that appear are:

\[
C^2 \begin{bmatrix} x_i \\ x_j \end{bmatrix} - C - C + C
\]
Potential Fix

• What if we add an appropriate multiple of

\[ x_{i1} x_{j1} \]

\[ x_{i2} x_{j2} \]

\[ x_{ir} x_{jr} \]

\[ x_{i1}' x_{j1}' \]

\[ x_{i2}' x_{j2}' \]

\[ x_{ir}' x_{jr}' \]

to our moment matrix?
Potential Fix Analysis

• This fix does work for $d = 4$ [HKPRS16]
• However, it seems rather ad-hoc.
• Remark: It is related to giving more weight to cliques which have more common neighbors, but that’s not quite what it does...
• Can we find a more principled general fix? Yes, see next lecture!
References


Appendix
Approximating $\tilde{E}[x_I]$ Calculation

- $\tilde{E}[x_I] = \frac{\binom{k}{|I|}}{\binom{d}{|I|}} \cdot \frac{N_d(I)}{N_d(\emptyset)}$

- If $I$ is a clique then $N_d(I) \approx 2^{\left(\binom{|I|}{2}\right) - \binom{d}{2}} \left(\frac{n - |I|}{d - |I|}\right)$

- As a special case, $N_d(\emptyset) \approx 2^{-\binom{d}{2}} \binom{n}{d}$

- If $I$ is a clique then

$$\tilde{E}[x_I] \approx \frac{\binom{k}{|I|} \cdot 2^{\left(\binom{|I|}{2}\right) - \binom{d}{2}} \left(\frac{n - |I|}{d - |I|}\right)}{\left(\frac{d}{|I|}\right)^2 \left(\frac{d}{2}\right) \binom{n}{d}} = 2^{\left(\binom{|I|}{2}\right)} \frac{\binom{k}{|I|}}{\binom{n}{|I|}} \approx 2^{\left(\binom{|I|}{2}\right)} \frac{k^{|I|}}{n^{|I|}}$$