

A Note on Property Testing Sum of Squares and Multivariate Polynomial Interpolation

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Abstract

In this paper, we investigate property testing whether or not a degree d multivariate polynomial is a sum of squares or is far from a sum of squares. We show that if we require that the property tester always accepts YES instances and uses random samples, $n^{\Omega(d)}$ samples are required, which is not much fewer than it would take to completely determine the polynomial. To prove this lower bound, we show that with high probability, multivariate polynomial interpolation matches arbitrary values on random points and the resulting polynomial has small norm. We then consider a particular polynomial which is non-negative yet not a sum of squares and use pseudo-expectation values to prove it is far from being a sum of squares.

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1 Introduction

In recent years, property testing and the sum of squares hierarchy have both been fruitful areas of research. In property testing, we aim to find algorithms which only look at a small portion of the input. However, instead of requiring an exact answer, we only require that we can distinguish between a function which has a given property and a function which is far from having that property. Thus far, property testers have been found for many properties of boolean functions including monotonicity, dictatorships, juntas, and being low degree [9, 16, 5]. For a survey on results in property testing, see Oded Goldreich's book [8].

The sum of squares hierarchy, independently investigated by Nesterov [14], Shor [19], Parrilo [17], and Lasserre [12], is a hierarchy of semidefinite programs which has the advantages of being broadly applicable, powerful, and in some sense, simple. The sum of squares hierarchy is broadly applicable because it can be applied to any system of polynomial equations over the reals and most problems of interest can be put into this form. The sum of squares hierarchy is surprisingly powerful; it captures the best known algorithms for several problems including the Goemans-Williamson algorithm for maximum cut [7], the Geomans-Linial relaxation for sparsest cut (analyzed by Arora, Rao, Vazirani [3]), and the subexponential time algorithm found by Arora, Barak, and Steurer [2] for unique games. Finally, the sum of squares hierarchy is in some sense simple as all that it uses is the fact that squares must be non-negative over the real numbers. For a survey on the sum of squares hierarchy, see Barak and Steurer's survey [1].

A central question in researching the sum of squares hierarchy is determining whether a given polynomial is non-negative and whether it is a sum of squares. In the setting where we know all the coefficients of the polynomial, we can determine whether it is a sum of squares in polynomial time using semidefinite programming while determining whether it is non-negative is NP-hard. In this paper, we consider the question of property testing whether a polynomial is a sum of squares on random samples. In this setting, rather than knowing the full polynomial, we only have its value on randomly sampled points. However, we only need to determine whether it is a sum of squares or is far from being a sum of squares.

This work is also related to research on the difference between non-negative polynomials and polynomials which are sum of squares. This research began with Hilbert [11], who proved the existence of polynomials which are non-negative yet not a sum of squares. The first explicit example of such a polynomial was found by Motzkin [13]. More recently, Bleckherman [6] showed that there are significantly more polynomials which are non-negative than polynomials which are sums of squares. That said, to the best of our knowledge these papers do not analyze the distance of these polynomials from being sums of squares.

1.1 Results and Outline

Our main result is the following

Theorem 1.1. *For all $d \geq 2$ and all $\delta > 0$, there is an $\epsilon > 0$ such that for sufficiently large n , if we require that our property tester always accepts YES instances and use random samples then property testing whether a degree $2d$ polynomial is a sum of squares requires at least $n^{\frac{d}{2}-\delta}$ samples.*

Along the way, we prove the following result for multivariate polynomial interpolation on random points:

Theorem 1.2. *For all d and all $\delta > 0$, given points $p_1, \dots, p_m \in \mathbb{R}^n$ randomly sampled from the multivariate normal distribution with covariance matrix Id , if n is sufficiently large and $m \leq n^{d-\delta}$ then with very high probability, for all v_1, \dots, v_m , there is a polynomial g of degree d such that*

1. $\forall i, g(p_i) = v_i$
2. $\|g\|$ is $O\left(\frac{\|v\|}{n^{\frac{d}{2}}}\right)$
3. If we further have that $m \leq n^{\frac{d}{2}-\delta}$ then $\|g^2\|$ is $\tilde{O}\left(\frac{\|v\|^4}{n^{2d}}\right)$

This paper is organized as follows. In Section 2 we give definitions and conventions which we will use for the remainder of the paper. In Section 3, as a warm-up we consider the question of property testing non-negativity. This question is non-trivial because of how distance is defined in our setting. In Section 4 we describe our tester for being a sum of squares which we will prove a lower bound against. In Section 5, we prove our theorem on multivariate polynomial interpolation, showing that this tester will accept with high probability as long as the values it receives are non-negative and not too large. Finally, in Section 6 we complete our lower bound by giving a non-negative function f of norm 1 and lower bounding its distance from being a sum of squares using pseudo-expectation values.

2 Preliminaries

For our results, we consider randomly sampling bounded degree real-valued polynomials over the multivariate normal distribution. We use the following conventions

1. We take d or $2d$ to be the degree of our polynomials and assume that d is a constant.
2. We take m to be the number of sampled points.
3. Often, we will not be precise with functions of d or logarithmic factors, so we absorb such functions into an \tilde{O} .

We use the following definitions on the multivariate normal distribution.

Definition 2.1.

1. $\mathcal{N}(0, 1)$ is the univariate normal distribution with probability density $\mu(x) = \frac{1}{\sqrt{2\pi}}e^{-\frac{x^2}{2}}$.
2. $\mathcal{N}(0, Id)$ is the multivariate normal distribution with probability density

$$\mu(x_1, \dots, x_n) = \frac{1}{(2\pi)^{\frac{n}{2}}}e^{-\frac{\|x\|^2}{2}} = \prod_{i=1}^n \mu(x_i)$$

3. For real-valued functions f, g , we define the inner product $\langle f, g \rangle = \int_{\mathcal{N}(0, Id)} fg$. We define $\|f\| = \sqrt{\langle f, f \rangle}$

4. Given a set of real valued functions S and a function f , we define the distance of f from S to be $d(f, S) = \min_{g: g \in S} \{\|f - g\|\}$

Remark 2.2. It should be noted that this definition of distance differs from the definition of distance commonly used in the property testing literature, which is $d(f, S) = \min_{g: g \in S} \{\mu(\{x : f(x) \neq g(x)\})\}$. We use this definition of distance as it is more suitable for analyzing polynomials; if two polynomials have almost identical coefficients they will be very close to each other under Definition 2.1 but will be distance 1 from each other using this definition.

To help us index monomials, we use the following definitions:

Definition 2.3.

1. We use I (and occasionally J) to denote a multi-set of elements in $[1, n]$.
2. We define $x_I = \prod_{i \in I} x_i$.
3. We define $|I|$ to be the total number of elements of I (counting multiplicities)
4. Given an I and a $k \in [1, n]$, we define I_k to be the multiplicity of k in I .
5. Given an I and a $t \in [1, |I|]$, we define $I(t)$ to be the number such that $\sum_{j=1}^{I(t)-1} I_j < t$ but $\sum_{j=1}^{I(t)} i_j \geq t$. In other words, if we put the elements of I in sorted order, $I(t)$ will be the t th element which appears.

Remark 2.4. Sometimes we will also attach subscripts to I . To distinguish between this and the notation above, we will only use the above notation with the letters k and t and in the case where both occur, we will put the subscript in parentheses on the inside. For example, if we want the multiplicity of k in I_j then we will write $(I_j)_k$

For our analysis, it will be extremely useful to work with the orthonormal basis of polynomials. For the multivariate normal distribution, this basis is the Hermite polynomials and we use the following definitions

Definition 2.5.

1. We define $h_j(x)$ to be the j th Hermite polynomial normalized so that $\|h_j\| = 1$
2. Given an I , we define $h_I(x_1, \dots, x_n) = \prod_{k=1}^n h_{I_k}(x_k)$

Proposition 2.6. The multivariate polynomials $h_I(x_1, \dots, x_n)$ are an orthonormal basis over $\mathcal{N}(0, Id)$.

Corollary 2.7 (Fourier decomposition and Parseval's theorem). For any polynomial g we can write $g = \sum_I c_I h_I$ where $c_I = \langle g, h_I \rangle$ and we have that $\|g\|^2 = \sum_I c_I^2$.

Remark 2.8. Our results can be generalized to different product distributions. To do this, simply replace the Hermite polynomials with the appropriate orthonormal basis in a single variable.

Finally, we need the concept of pseudo-expectation values, which is extremely useful for analyzing the sum of squares hierarchy. As we show below, pseudo-expectation values allow us to lower bound the distance of a degree $2d$ polynomial f from being a sum of squares.

Definition 2.9. We define degree $2d$ pseudo-expectation values to be a linear map \tilde{E} from polynomials of degree at most $2d$ to \mathbb{R} which satisfies the following conditions:

1. $\tilde{E}[1] = 1$
2. $\forall g : \deg(g) \leq d, \tilde{E}[g^2] \geq 0$

This second condition can be equivalently stated as follows

Definition 2.10. Given degree $2d$ pseudo-expectation values \tilde{E} , define the moment matrix M to be the matrix with rows and columns indexed by monomials $\{x_I : |I| \leq d\}$ and entries $M_{IJ} = \tilde{E}[x_I x_J]$

Proposition 2.11. The condition that $\forall g : \deg(g) \leq d, \tilde{E}[g^2] \geq 0$ is equivalent to the condition that $M \succeq 0$.

We now show how pseudo-expectation values can be used to show a lower bound on how far a polynomial f is from being a sum of squares of degree $\leq 2d$.

Lemma 2.12. Given pseudo-expectation values \tilde{E} , if $\tilde{E}[f] < 0$ then for all g of degree at most $2d$ such that g is a sum of squares, $\|f - g\|^2 \geq \frac{(\tilde{E}[f])^2}{\sum_{I:|I|\leq 2d} (\tilde{E}[h_I])^2}$

Proof. Write $g - f = \sum_I c_I h_I$. Observe that $\tilde{E}[g - f] = -\tilde{E}[f] + \tilde{E}[g] \geq -\tilde{E}[f]$ which implies that

$$\sum_I c_I \tilde{E}[h_I] \geq -\tilde{E}[f]$$

Using Cauchy-Schwarz,

$$\sqrt{\sum_I c_I^2} \sqrt{\sum_I (\tilde{E}[h_I])^2} \geq \sum_I c_I \tilde{E}[h_I] \geq -\tilde{E}[f]$$

Since both sides are non-negative, the result follows by squaring both sides and dividing both sides by $\sum_I (\tilde{E}[h_I])^2$. \square

3 Property testing non-negativity of degree d polynomials

As a warm-up, in this section we consider the closely related question of property testing whether a degree d polynomial is non-negative or far from being non-negative. While this question is trivial under the definition of distance in Remark 2.2, it is non-trivial with our norm-based definition of distance. We also note that to the best of our knowledge, this problem is open if we consider the distance from the smaller set of non-negative degree d polynomials rather than the set of all non-negative functions.

Theorem 3.1. *The following property tester distinguishes with high probability between an f which is a degree d non-negative polynomial and an f which is a degree d polynomial that is $(\epsilon \|f\|)$ -far from being non-negative.*

1. Take $\frac{10B}{\epsilon}$ random samples where $B = e \left(4 + \ln \left(\frac{1}{\epsilon}\right)\right)^{2d}$
2. If any sample gives a negative value, return NO. Otherwise, return YES.

Proof. Normalize f so that $\|f\| = 1$. Let f^- be the negative part of f and let f^+ be the non-negative part of f . If f is ϵ -far from being non-negative yet $\frac{10B}{\epsilon}$ random samples fails to find a negative value of f with high probability then we must have that $\|f^-\| > \epsilon$ yet f^- is supported on a set of measure at most $\frac{\epsilon}{B}$.

However, by a corollary of the hypercontractivity theorem (which applies in the Gaussian setting as well, see O'Donnell's lecture notes on hypercontractivity [15]), for all q ,

$$\|f^-\|_q \leq \|f\|_q \leq (\sqrt{q-1})^d \|f\|_2 = (\sqrt{q-1})^d$$

Given that $\|f^-\|_2 \geq \epsilon$ and f^- is supported on a set of measure at most $\frac{\epsilon}{B}$, for $q > 2$ we minimize $\|f^-\|_q$ (over all functions, not just polynomials) by setting f^- equal to $-\sqrt{B}$ on a set of measure $\frac{\epsilon}{B}$ and setting $f^- = 0$ elsewhere. This implies that

$$\epsilon B^{\frac{q}{2}-1} \leq \|f^-\|_q^q < q^{\frac{dq}{2}}$$

This gives a contradiction when $B \geq \left(\frac{1}{\epsilon} q^{\frac{dq}{2}}\right)^{\frac{2}{q-2}}$. Taking $q = 4 + \ln \left(\frac{1}{\epsilon}\right)$,

$$\left(\frac{1}{\epsilon} q^{\frac{dq}{2}}\right)^{\frac{2}{q-2}} \leq \left(\frac{1}{\epsilon}\right)^{\frac{1}{1+\ln\left(\frac{1}{\epsilon}\right)}} \left(4 + \ln \left(\frac{1}{\epsilon}\right)\right)^{2d} \leq e \left(4 + \ln \left(\frac{1}{\epsilon}\right)\right)^{2d}$$

Thus, we have a contradiction as long as $B \geq e \left(4 + \ln \left(\frac{1}{\epsilon}\right)\right)^{2d}$. □

If we instead consider the distance from non-negative degree d polynomials, it is no longer clear whether any degree d f which is far from being a non-negative degree d polynomial must be negative on a constant proportion of the inputs. We leave this as a question for further research.

4 Algorithm for testing SOS

In this section, we describe a tester for property testing whether a polynomial f of degree $2d$ is a sum of squares of norm at most 1 or is far from being a degree $2d$ sum of squares of norm at most 1. This tester is optimal over all testers which always accept YES instances. Thus, to prove our lower bound it is sufficient to show that this tester fails with high probability.

Given data $\{f(p_i) = v_i, i \in [1, m]\}$, we can try to test whether a polynomial f of degree at most $2d$ is a sum of squares as follows.

Definition 4.1. *Given a coefficient matrix M with rows and columns indexed by multi-sets I of size at most d , define $f_M = \sum_J \left(\sum_{I, I': I \cup I' = J} M_{II'}\right) x_J$.*

Proposition 4.2. *A polynomial f can be written as a sum of squares if and only there exists a coefficient matrix M such that $f_M = f$ and $M \succeq 0$.*

Thus, we can search for a coefficient matrix M such that

1. $\forall i \in [1, m], f_M(x_i) = v_i$
2. $M \succeq 0$

If such a coefficient matrix M is found then we output YES, otherwise we output NO.

This algorithm outputs YES precisely when there is polynomial f_M of degree at most $2d$ which is a sum of squares and matches the data. However, for all we know, $\|f_M\|$ could be very high. On the other hand, in multivariate polynomial interpolation, when the polynomial is underdetermined it is natural to minimize the norm of the polynomial. To take this into account, we instead consider the following property testing problem and algorithm:

Assumption: One of the following cases holds:

1. f has degree at most $2d$, f is a sum of squares, and $\|f\| \leq 1$.
2. For all g such that g has degree at most $2d$, g is a sum of squares, and $\|g\| \leq 1$, $\|f - g\| > \epsilon$.

Algorithm: Search for a coefficient matrix satisfying the following conditions:

1. $\forall i \in [1, m], f_M(x_i) = v_i$
2. $\|f_M\| \leq 1$
3. $M \succeq 0$

Remark 4.3. *These conditions on M are all convex, so this algorithm can be implemented with convex optimization.*

Remark 4.4. *This algorithm is optimal if we require that the property tester always accept YES instances, as it says YES precisely when there is a function f_M which is a sum of squares, matches the data, and has norm at most 1.*

To prove our lower bound, it is necessary and sufficient to find a degree $2d$ polynomial f of norm 1 which is (ϵ) -far from being a degree $2d$ sum of squares such that if we take m randomly sampled points where $m \leq n^{\frac{d}{2}-\delta}$, this tester accepts f with high probability.

5 Norm Bounds for Multivariate Polynomial Interpolation

In polynomial interpolation, we are given points p_1, \dots, p_m and values v_1, \dots, v_m and we want to find a polynomial g of a given degree d such that $\forall i, g(p_i) = v_i$. Single variable polynomial interpolation is very well understood; it can be achieved precisely when $m \leq d + 1$. However, multivariable polynomial interpolation is much less well understood. In this section, we consider the case when the p_i are random. In this case, interpolation is almost surely possible as long as $m \leq \sum_{i=0}^d \binom{n+i-1}{i}$, where n is the number of variables. However, this does not say anything about the norm of the resulting polynomial. In this section, we show that for all $\delta > 0$, if $m \leq n^{d-\delta}$ and n is sufficiently large then we can find a g which matches all the data and has small norm. Moreover, if $m \leq n^{\frac{d}{2}-\delta}$ then $\|g^2\|$ has small norm as well. More precisely, we show the following theorem.

Definition 5.1. We define C to be the expected value of $\sum_{I:0<|I|\leq d} h_I(p)^2$ for a random point p .

Theorem 5.2. For all d and all $\delta > 0$, given points $p_1, \dots, p_m \in \mathbb{R}^n$ randomly sampled from the multivariate normal distribution with covariance matrix Id , if n is sufficiently large and $m \leq n^{d-\delta}$ then with very high probability, for all v_1, \dots, v_m , there is a polynomial g of degree d such that

1. $\forall i, g(p_i) = v_i$
2. $\|g\| = (1 \pm o(1)) \frac{\|v\|}{\sqrt{C}}$
3. If we further have that $m \leq n^{\frac{d}{2}-\delta}$ then $\|g^2\|$ is $\tilde{O}(\frac{\|v\|^4}{n^{2d}})$

This theorem shows that our tester will accept with high probability as long as $m \leq n^{\frac{d}{2}-\delta}$ and all our sampled points have non-negative values. To see this, note that given data $f(p_i) = v_i$, this theorem says that with high probability there is a g such that $g(p_i) = \sqrt{v_i}$ and $\|g^2\|$ has small norm. Thus, g^2 matches the data and has small norm so the tester must accept.

5.1 Construction of the function g

To construct our function g , we use the following strategy:

1. We construct a function g_i of degree d for each point p_i .
2. We take the matrix M where $M_{ij} = g_j(p_i)$.
3. We take x to be a solution to $Mx = v$.
4. We take $g = \sum_j x_j g_j$.

Proposition 5.3. For all i , $g(p_i) = v_i$.

Proof. For all i , $g(p_i) = \sum_j x_j g_j(p_i) = \sum_j M_{ij} x_j = v_i$. □

We now construct the functions g_i of degree d for each point p_i . These functions are constructed so that with high probability, for all i , $g_i(p_i) \approx 1$ and for all $i \neq j$, $|g_i(p_j)|$ is small.

Definition 5.4. Given a point $p_i = (v_1, \dots, v_n)$, we define $g_i = \frac{\sum_{I:0<|I|\leq d} h_I(p_i) h_I}{C}$

5.2 Analysis of the function g

To analyze the function g , it is useful to consider the following matrix H which is closely related to M .

Definition 5.5. We define H to be the matrix with rows indexed by I where $0 < |I| \leq d$, columns indexed by i , and entries $H_{Ii} = h_I(p_i)$.

Lemma 5.6. $M = \frac{H^T H}{C}$

Proof. Observe that

$$\frac{1}{C}(H^T H)_{ij} = \frac{1}{C} \sum_{I:0<|I|\leq d} (h_I(p_i)h_I(p_j)) = \left(\frac{\sum_{I:0<|I|\leq d} h_I(p_j)h_I(p_i)}{C} \right) (p_i) = g_j(p_i) = M_{ij}$$

□

Proposition 5.7. $g = \frac{1}{C} \sum_I (Hv)_I h_I$

We now have that

$$\|g\|^2 = \frac{1}{C^2} \sum_I \left(\sum_{j=1}^m x_j H_{Ij} \right)^2 = \frac{1}{C^2} x^T H^T H x = \frac{1}{C} x^T M x = \frac{1}{C} v^T M^{-1} v$$

In the next subsection, we will show that with high probability M is very close to the identity which immediately implies that with high probability, $\|g\| = (1 \pm o(1)) \frac{\|v\|}{\sqrt{C}}$, as needed.

5.3 Analysis of H and M

In this subsection, we analyze the matrices H and M . We begin by analyzing H in order to develop the necessary techniques.

Theorem 5.8. *For all $\delta > 0$ and all d , if n is sufficiently large and $m \leq n^{d-\epsilon}$ then with high probability, $\|H\|$ is $\tilde{O}(n^{\frac{d}{2}})$*

Proof. We can use the trace power method to probabilistically bound $\|H\|$. For this, we need to bound

$$E \left[\text{tr}((HH^T)^q) \right] = \sum_{i_1, \dots, i_q, I_1, \dots, I_q: \forall j, 0 < |I_j| \leq d} E \left[\prod_{j=1}^q H_{I_j i_j} H_{I_{j+1} i_{j+1}} \right]$$

where we take $I_{q+1} = I_1$ and $i_{q+1} = i_1$. We partition this sum based on the intersection pattern P of which of the i_1, \dots, i_q are equal to each other and how I_1, \dots, I_q interact with each other. We then analyze which intersection patterns give terms with nonzero expected value.

Definition 5.9. *We define an intersection pattern P to be the following data:*

1. *For all $j' \neq j$, P has the equality $i_{j'} = i_j$ or the inequality $i_{j'} \neq i_j$*
2. *For all j, j', t, t' , P has the equality $I_{j'}(t') = I_j(t)$ or the inequality $I_{j'}(t') \neq I_j(t)$*

where these equalities and inequalities are consistent with each other (i.e. transitivity is satisfied for the equalities).

Lemma 5.10. *There are at most $(4dq)^{4dq}$ possible intersection patterns.*

Proof. Choose an arbitrary ordering of the i_j and an arbitrary ordering of the $I_j(t)$. To specify an intersection pattern, it suffices to specify which i_j and $I_j(t)$ are equal to previous i_j and $I_j(t)$ and if so, to specify one of the equalities which hold. The total number of choices is at most $(4dq)^{4dq}$. □

Lemma 5.11. For any intersection pattern P which gives a nonzero expected value, letting $x = |\{k : \exists j : (I_j)_k > 0\}|$ and letting y be the number of distinct i_j , $y + \frac{x}{d} \leq q + 1$

Proof. The key observation is that if we consider the multiset

$$\{(i_j, k) : (I_j)_k > 0\} \cup \{(i_j, k) : (I_{j+1})_k > 0\} = \{(i_j, k) : (I_j)_k > 0\} \cup \{(i_{j-1}, k) : (I_j)_k > 0\},$$

if any element of this multiset appears exactly once then $E \left[\prod_{j=1}^q H_{I_j i_j} H_{I_{j+1} i_j} \right] = 0$ over the random choices for the points $\{p_i\}$.

With this observation in mind, for each k , consider the graph formed by the edges $\{(i_{j-1}, i_j) : (I_j)_k > 0\}$. In a term with nonzero expectation, for all k , every vertex in this graph with nonzero degree must have degree at least 2 (where we consider loops as adding 2 to the degree). Thus, these graphs must consist of loops/cycles and loops/cycles joined by paths. This implies the following upper bound on x

Definition 5.12. Let G_y be the multi-graph consisting of the q edges $\{i_{j-1}, i_j\}$

Definition 5.13. Given a multi-graph G , we define $w(G)$ to be the maximum number such that $\exists G_1, \dots, G_t$ and w_1, \dots, w_t satisfying the following conditions

1. $w(G) = \sum_i w_i$
2. $\forall i, V(G_i) = V(G), E(G_i) \subseteq E(G), E(G_i)$ is nonempty, and no vertex of G_i has degree exactly 1 (where we consider loops as adding 2 to the degree).
3. $\forall i, w_i \geq 0$
4. $\forall j, \sum_{i: (i_{j-1}, i_j) \in E(G_i)} w_i \leq 1$

Lemma 5.14. For any intersection pattern which gives a nonzero expected value, $x = |\{k : \exists j : (I_j)_k > 0\}| \leq d \cdot w(G_y)$

Proof. For each $k : \exists j : (I_j)_k > 0$, we construct the graph G_k where $V(G_k) = V(G_y), E(G_k) = \{(i_{j-1}, i_j) : (I_j)_k > 0\}$ and assign it weight $\frac{1}{d}$. From the above observation, no vertex of G_k can have degree exactly 1 (where we consider loops as adding 2 to the degree). Also, we have that the total weight on any edge (i_{j-1}, i_j) is at most 1 as at most d graphs G_k contribute to it and each contribution is $\frac{1}{d}$. Thus, $\sum_{k: \exists j: (I_j)_k > 0} \frac{1}{d} = \frac{x}{d} \leq w(G_y)$, as needed. \square

With this bound in mind, we now prove the following lemma which will immediately imply our result.

Lemma 5.15. For all connected multi-graphs G , $w(G) + |V(G)| \leq |E(G)| + 1$

Proof. We first reduce to the case where every non-loop edge of G has multiplicity at least two with the following lemma.

Lemma 5.16. If G is a multigraph which has a non-loop edge e that appears with multiplicity 1 and G' is the graph formed by contracting this edge then $w(G) \leq w(G')$

Proof. Observe that if subgraphs G_1, \dots, G_t of G all have no vertex of degree exactly 1, then letting G'_1, \dots, G'_t be the graphs G_1, \dots, G_t formed by making the two endpoints of e equal and removing e (if present), G'_1, \dots, G'_t are subgraphs of G' and have no vertices of degree exactly 1. To see this, note that for any vertex v' in G'_i except for the vertex formed by making the two endpoints of e equal, the number of edges incident to v' is unaffected. For the v' formed by making the two endpoints of e equal, each of these endpoints must have had an edge besides e incident with it, so the degree of this v' is at least 2. \square

Using this lemma, if G has a non-loop edge e which appears with multiplicity 1, G' is the graph formed by contracting this edge, and $w(G') + |V(G')| \leq |E(G')| + 1$ then

$$w(G) + |V(G)| \leq w(G') + |V(G')| + 1 \leq |E(G')| + 2 = |E(G)| + 1$$

Thus, it is sufficient to prove the lemma for G' . Applying this logic repeatedly, it is sufficient to prove the result for the case where every non-loop edge of G has multiplicity at least two.

Definition 5.17. We define $E_{loop}(G)$ to be the multi-set of loops in G and we define $E_{nonloop}(G)$ to be $E(G) \setminus E_{loop}(G)$.

Lemma 5.18. For all G , $w(G) \leq |E_{loop}(G)| + \frac{|E_{nonloop}(G)|}{2}$

Proof. Let G_1, \dots, G_k and w_1, \dots, w_k be graphs and weights such that

1. $w(G) = \sum_{i=1}^k w_i$
2. $\forall i, V(G_i) = V(G), E(G_i) \subseteq E(G), E(G_i)$ is nonempty, and no vertex of G_i has degree exactly 1 (where we consider loops as adding 2 to the degree).
3. $\forall i, w_i \geq 0$
4. $\forall j, \sum_{i \in [1, k]: (i_{j-1}, i_j) \in E(G_i)} w_i \leq 1$

Observe that each G_i must either have at least one loop or at least two non-loop edges. Thus,

$$\begin{aligned} |E_{loop}(G)| + \frac{|E_{nonloop}(G)|}{2} &\geq \sum_{j: (i_{j-1}, i_j) \in E_{loop}(G)} \sum_{i: (i_{j-1}, i_j) \in E(G_i)} w_i + \frac{1}{2} \sum_{j: (i_{j-1}, i_j) \in E_{nonloop}(G)} \sum_{i: (i_{j-1}, i_j) \in E(G_i)} w_i \\ &= \sum_i \left(\sum_{j: (i_{j-1}, i_j) \in E(G_i) \cap E_{loop}(G)} w_i + \frac{1}{2} \sum_{j: (i_{j-1}, i_j) \in E(G_i) \cap E_{nonloop}(G)} w_i \right) \\ &\geq \sum_i w_i = w(G) \end{aligned}$$

as needed. \square

Lemma 5.19. If G is connected and every non-loop edge of G has multiplicity at least 2 then $|V(G)| \leq \frac{|E_{nonloop}(G)|}{2} + 1$

Proof. Imagine building up G from one isolated vertex. We can add loops for free, but every time we add a neighbor of an existing vertex, we must add at least two edges (as all edges have multiplicity at least two). \square

Putting these lemmas together, $|V(G)| + w(G) \leq |E_{loop}(G)| + |E_{nonloop}(G)| + 1 = |E(G)| + 1$ which proves Lemma 5.15. \square

Putting Lemmas 5.14 and 5.15 together, Lemma 5.11 follows immediately. We have that $y + \frac{x}{d} \leq |V(G)| + w(G) \leq |E(G)| + 1 = q + 1$, as needed. \square

We now consider the expression

$$E \left[\text{tr}((HH^T)^q) \right] = \sum_{i_1, \dots, i_q, I_1, \dots, I_q: \forall j, 0 < \deg(I_j) \leq d} E \left[\prod_{j=1}^q H_{I_j i_j} H_{I_{j+1} i_j} \right]$$

Lemma 5.11 implies that for any intersection pattern which gives a nonzero expected value, there are at most $\max_{x, y: y \geq 1, y + \frac{x}{d} \leq q+1} \{m^y n^x\} \leq mn^{dq}$ choices for $i_1, \dots, i_q, I_1, \dots, I_q$. By Lemma 5.10, there are at most $(4dq)^{4dq}$ possible intersection patterns. To complete our upper bound, we just need to show a bound on $E \left[\prod_{j=1}^q H_{I_j i_j} H_{I_{j+1} i_j} \right]$ for a particular $i_1, \dots, i_q, I_1, \dots, I_q$, which we do with the following lemma

Lemma 5.20. *For any $i_1, \dots, i_q, I_1, \dots, I_q$,*

$$E \left[\prod_{j=1}^q H_{I_j i_j} H_{I_{j+1} i_j} \right] \leq (4dq)^{4dq}$$

Proof. Observe that for any $i_1, \dots, i_q, I_1, \dots, I_q$, $E \left[\prod_{j=1}^q H_{I_j i_j} H_{I_{j+1} i_j} \right]$ is a product of expressions of the form $E \left[\left(\prod_{i=1}^k h_{j_i} \right) (x) \right]$ for some j_1, \dots, j_k . For each such expression we have the following bound.

Lemma 5.21. *Letting $d' = \sum_{i=1}^k j_i$, $E \left[\left(\prod_{i=1}^k h_{j_i} \right) (x) \right] \leq (d')^{2d'}$*

Proof. We use the fact that for all $j \geq 1$, the sum of the absolute values of the coefficients of h_j is at most j^j . This implies that the sum of the absolute values of the coefficients of $\prod_{i=1}^k h_{j_i}$ is at most $d'^{d'}$. Over a normal distribution $E[x^p] = \prod_{i=1}^{\frac{p}{2}} (2i - 1) \leq p^p$ if p is even and is 0 if p is odd, which implies the result. \square

The total sum of all the degrees is at most $2dq$ as this is the maximum number of pairs $I_j(t), i_j$ and $I_{j+1}(t), i_j$. Thus, the product over all of the expressions which we have is at most $(4dq)^{4dq}$, as needed. \square

Putting everything together,

$$E \left[\text{tr}((HH^T)^q) \right] = \sum_{i_1, \dots, i_q, I_1, \dots, I_q: \forall j, 0 < \deg(I_j) \leq \frac{d}{2}} E \left[\prod_{j=1}^q H_{I_j i_j} H_{I_{j+1} i_j} \right] \leq (4dq)^{8dq} mn^{dq}$$

We now apply Markov's inequality. For all q and all $\beta \geq 0$,

$$\begin{aligned} \Pr \left[\|H\| \geq \sqrt[2q]{n^\beta E \left[\text{tr}((HH^T)^q) \right]} \right] &= \Pr \left[\|H\|^{2q} \geq n^\beta E \left[\text{tr}((HH^T)^q) \right] \right] \\ &\geq \Pr \left[\text{tr}((HH^T)^q) \geq n^\beta E \left[\text{tr}((HH^T)^q) \right] \right] \leq \frac{1}{n^\beta} \end{aligned}$$

Applying this with $q \sim dq\beta \log n$, Theorem 5.8 follows. \square

With the techniques we developed to prove Theorem 5.8, we can now analyze M .

Theorem 5.22. *For all d and all $\delta > 0$, for sufficiently large n , if $m \leq n^{d-\delta}$ and we write $M = Id + M'$ then with high probability $\|M'\| \ll 1$.*

Proof sketch: This theorem can be proved by considering the diagonal part and off-diagonal part of M . For the diagonal part of M , observe that $M_{ii} = \frac{1}{C} \sum_{I:0<|I|\leq d} h_I(p_i)^2$. Since C is the expected value of $\sum_{I:0<|I|\leq d} h_I(p)^2$ for a random point p and this value is tightly concentrated around its expectation, with high probability M_{ii} will be $1 \pm o(1)$ for all i . For the off-diagonal part of M , we can use the trace power method to bound its norm. Let M'' be the off-diagonal part of M .

Lemma 5.23. *For all d and all $\delta > 0$, for sufficiently large n , if $m \leq n^{d-\delta}$ then with high probability $\|M''\|$ is $\tilde{O}\left(\frac{\sqrt{m}}{n^{\frac{d}{2}}}\right)$*

Proof sketch: Observe that

$$E[\text{tr}((M'')^q)] = \frac{1}{C^q} \sum_{i_1, \dots, i_q, I_1, \dots, I_q: \forall j, 0 < \deg(I_j) \leq \frac{d}{2}, i_j \neq i_{j+1}} E \left[\prod_{j=1}^q H_{I_j i_j} H_{I_{j+1} i_j} \right]$$

Up to the $\frac{1}{C^q}$ factor, this is the same expression we had for $E[\text{tr}((HH^T)^q)]$ except that since we are restricting ourselves to the off-diagonal part of M we additionally have the constraint that $i_j \neq i_{j+1}$ for all j . This constraint implies that for any term with nonzero expected value, there is no k such that $(I_j)_k > 0$ and $(I_{j'})_k = 0$ for all $j \neq j'$. This in turn implies that we only need to consider intersection patterns with $x \leq \frac{dq}{2}$, which means that the maximum number of choices for a given intersection pattern is at most $(m^{\frac{q}{2}+1} n^{\frac{dq}{2}})$ rather than mn^{dq} . Thus, our final bound on $E[\text{tr}((M'')^q)]$ will be

$$E[\text{tr}((M'')^q)] \leq (4dq)^{8dq} \frac{1}{C^q} m^{\frac{q}{2}+1} n^{\frac{dq}{2}}$$

Recalling that $C = \sum_{I:0<|I|\leq d} h_I(p)^2$, C is $\Theta(n^d)$ and the result can be shown in same the way as Theorem 5.8 using Markov's inequality (where we choose an even q). \square

\square

Remark 5.24. *In fact, our analysis of M gives us improved norm bounds on $\|H\|$. In particular, with high probability $\|H\|$ is $\sqrt{C}(1 \pm o(1))$*

5.4 Analysis of $\|g^2\|$

In this subsection, we show how to probabilistically bound $\|g^2\|$.

Definition 5.25. *We define the matrix Q so that $Q_{J(I,I')}$ is the coefficient of h_J in $h_I h_{I'}$.*

We have that

$$g^2 = \frac{1}{C^2} \sum_J \left(Q_{J(I,I')} \sum_{I,I'} \sum_{j=1}^m \sum_{j'=1}^m x_j H_{I_j} x_{j'} H_{I'_{j'}} \right) h_J$$

Thus,

$$\begin{aligned}\|g^2\|^2 &= \frac{1}{C^4} \sum_J \left(Q_{J(I,I')} \sum_{I,I'} \sum_{j=1}^m \sum_{j'=1}^m x_j H_{I_j} x_{j'} H_{I'_{j'}} \right)^2 \\ &= \frac{1}{C^4} (x \otimes x)^T ((H \otimes H)^T Q^T Q (H \otimes H)) (x \otimes x)\end{aligned}$$

Theorem 5.26. *For all d, δ and all sufficiently large n , if $m \leq n^{\frac{d}{2}-\delta}$ then with high probability, for all vectors x , $\|Q(H \otimes H)(x \otimes x)\|^2$ is $\tilde{O}(n^{2d}\|x\|^4)$*

Proof sketch. We break $Q(H \otimes H)$ into two parts.

1. Let A be the matrix such that $A_{J(j,j')} = (Q(H \otimes H))_{J(j,j')}$ if $j' = j$ and is 0 otherwise.
2. Let R be the matrix such that $R_{J(j,j')} = (Q(H \otimes H))_{J(j,j')}$ if $j' \neq j$ and is 0 otherwise.

For the first part, we observe that letting $A_{(j,j)}$ be the (j, j) column of A ,

$$\|A(x \otimes x)\| = \left\| \sum_j x_j^2 A_{jj} \right\| \leq \left(\sum_j x_j^2 \right) \max_j \{ \|A_{jj}\| \} = \|x\|^2 \max_j \{ \|A_{jj}\| \}$$

Thus, $\|A(x \otimes x)\|^2 \leq (\max_j \{ \|A_{jj}\| \})^2 \|x\|^4$ and it is sufficient to probabilistically bound $\max_j \{ \|A_{jj}\| \}$

Lemma 5.27. *With high probability, $\max_j \{ \|A_{jj}\| \}$ is $\tilde{O}(n^d)$*

Proof. Observe that for all j ,

$$\|A_{jj}\|^2 = \sum_{J, I_1, I_2, I_3, I_4} Q_{J(I_1, I_2)} Q_{J(I_3, I_4)} H_{I_1 j} H_{I_2 j} H_{I_3 j} H_{I_4 j}$$

The entries of Q are $O(1)$ and with high probability the entries of H are $\tilde{O}(1)$, so we just need to bound the number of I_1, I_2, I_3, I_4 which give a nonzero contribution. For this, observe that for any nonzero term,

1. $I_1 \Delta I_2 \subseteq J$ and $I_3 \Delta I_4 \subseteq J$ where Δ is the symmetric difference.
2. $J \subseteq I_1 \cup I_2$ and $J \subseteq I_3 \cup I_4$

Together, these observations imply that there cannot be a k such that precisely one of $(I_1)_k, (I_2)_k, (I_3)_k, (I_4)_k$ is nonzero. In turn, this implies that there are $O(n^{2d})$ choices for I_1, I_2, I_3, I_4 which give a nonzero contribution and the result follows. \square

For the second part, we bound the norm of R .

Lemma 5.28. *For all d, δ and all sufficiently large n , if $m \leq n^{\frac{d}{2}-\delta}$ then with high probability, $\|R\|$ is $\tilde{O}(n^d)$.*

Proof sketch. This can be shown using the trace power method. We have that

$$E \left[(R^T R)^q \right] = \sum_{\{j_1, j'_1, J_1, I_{11}, I_{12}, I_{13}, I_{14}, \dots, j_q, j'_q, J_q, I_{q1}, I_{q2}, I_{q3}, I_{q4}\}} E \left[\prod_{a=1}^q Q_{J_a(I_{a1}, I_{a2})} Q_{J_a(I_{a3}, I_{a4})} H_{I_{a1}j_a} H_{I_{a2}j'_a} H_{I_{a3}j_{a+1}} H_{I_{a4}j'_{a+1}} \right]$$

Similar to before, we can partition this sum into intersection patterns and consider which patterns have nonzero expectation.

Definition 5.29. We take x to be the number of distinct k such that $(I_{ai})_k > 0$ for some a, i and we take y to be the number of distinct j_a and j'_a .

Lemma 5.30. For any term with nonzero expected value, $y + \frac{2x}{d} \leq 4q + 2$

Proof sketch. In any term with nonzero expected value, following the same logic as before, for each block $I_{a1}, I_{a2}, I_{a3}, I_{a4}$ there cannot be a k such that precisely one of $(I_{a1})_k, (I_{a2})_k, (I_{a3})_k, (I_{a4})_k$ is nonzero. If every k which appears in a block appears in at least two blocks then $x \leq d$. We trivially have that $y \leq 2q$ so the result holds in this case.

If there is a k which appears in only one block then this implies an equality between $j_a = j_{a+1}$ or an equality $j'_a = j'_{a+1}$. Roughly speaking, each such equality allows d additional values k to only appear in one block, decreasing y by 1 but increasing x by $\frac{d}{2}$. This leaves $y + \frac{2x}{d}$ unchanged. To see why we have the $+2$, consider the extreme case when all the j_a are equal and all the j'_a are equal. In this case $y = 2$ and we can have $x = 2dq$. \square

With this lemma in hand, since $m \leq n^{\frac{d}{2}-\delta}$, for any intersection pattern which gives a nonzero expected value, the total number of choices for the $j_a, j'_a, I_{a1}, I_{a2}, I_{a3}, I_{a4}$ is $O(m^2 n^{dq})$. Lemma 5.28 can now be shown using the same techniques used to prove Theorem 5.8. \square

Putting these results together, it follows that with high probability, for all vectors x , $\|Q(H \otimes H)(x \otimes x)\|^2$ is $\tilde{O}(n^{2d} \|x\|^4)$, as needed.

Remark 5.31. While Lemma 5.28 is essentially tight, it should be possible to obtain the same bound on $\|g^2\|^2$ for $m \leq n^{d-\delta}$ if we can effectively use the fact that we are dealing with $(x \otimes x)$ rather than an arbitrary vector, just as we did for A . We leave this as a question for further research. \square

6 A non-negative polynomial which is far from being a sum of squares

In this section, we complete our lower bound by giving a non-negative polynomial f and showing that f is far from being a sum of squares.

Definition 6.1. We take $f = \frac{r}{2}x^{r+2}y^r + \frac{r}{2}x^r y^{r+2} - (r+1)x^r y^r + (1+c)$ where $c \geq 0, r \geq 2$ is even, and we take $d = 2r + 2$

Remark 6.2. This polynomial f is a generalization of the Motzkin polynomial (which is the case $r = 2, c = 0$).

Lemma 6.3. $f \geq c$

Proof. Observe that $(x^{r+2}y^r)^{\frac{r}{2r+2}}(x^r y^{r+2})^{\frac{r}{2r+2}}(1)^{\frac{2}{2r+2}} = x^r y^r$ and $\frac{r}{2r+2} + \frac{r}{2r+2} + \frac{2}{2r+2} = 1$. By the AM-GM inequality, $\frac{r}{2r+2}(x^{r+2}y^r) + \frac{r}{2r+2}(x^r y^{r+2}) + \frac{2}{2r+2} \geq x^r y^r$ and the result follows. \square

Theorem 6.4. f is $\left(\frac{1}{(d^3 \sqrt[2]{2+c})^{2d^4}}\right)$ -far from being SOS.

Proof. We take the following pseudo-expectation values.

Definition 6.5. Take $k > 1$ and take $B = (kd^3)^{(3d^3)}$. We split up the pseudo-expectation values $\tilde{E}[x^a y^b]$ into cases as follows

1. If $a > b$ then we take $\tilde{E}[x^a y^b] = \frac{(kd^3)^{(a^2+(a+b)^2)}}{(kd^3)^{2d^2}} B^{a-\frac{r+2}{r}b}$.
2. If $b > a$ then we take $\tilde{E}[x^a y^b] = \frac{(kd^3)^{(b^2+(a+b)^2)}}{(kd^3)^{2d^2}} B^{a-\frac{r+2}{r}b}$.
3. For all $a > 0$ we take $\tilde{E}[x^a y^a] = \frac{4(a^2)}{4(r^2)} k^a$
4. We have $\tilde{E}[1] = 1$.
5. We take $\tilde{E}[p(x, y)q(\text{other variables})] = \tilde{E}[p(x, y)]E[q]$. This guarantees that $\tilde{E}[h_I] = 0$ whenever I contains a variable besides x and y .

Proposition 6.6. $\tilde{E}[f] = c + (r+1)(1-k^r)$

Proof. This follows immediately from the observations that $\tilde{E}[x^{r+2}y^r] = \tilde{E}[x^r y^{r+2}] = 1$ and $\tilde{E}[x^r y^r] = k^r$. \square

We need to show that these pseudo-expectation values give a PSD moment matrix.

Proposition 6.7. For all a, b such that $a + b \leq d$,

$$\tilde{E}[x^a y^b] \geq \min \left\{ \frac{(kd^3)^{(a^2+(a+b)^2)}}{(kd^3)^{2d^2}} B^{a-\frac{r+2}{r}b}, \frac{(kd^3)^{(b^2+(a+b)^2)}}{(kd^3)^{2d^2}} B^{a-\frac{r+2}{r}b} \right\}$$

Lemma 6.8. For all a_1, b_1, a_2, b_2 such that $a_1 + b_1 \leq d, a_1 + b_2 \leq d, a_1 \neq a_2$ or $b_1 \neq b_2$, and $a_1 \neq b_1$ or $a_2 \neq b_2$,

$$\tilde{E}[x^{a_1+a_2} y^{b_1+b_2}] \leq \frac{1}{2d^2} \sqrt{\tilde{E}[x^{2a_1} y^{2b_1}] \tilde{E}[x^{2a_2} y^{2b_2}]}$$

Proof. If $a_1 + a_2 > b_1 + b_2$ then we have that

$$\sqrt{\tilde{E}[x^{2a_1} y^{2b_1}] \tilde{E}[x^{2a_2} y^{2b_2}]} \geq \frac{(kd^3)^{(2a_1^2+2(a_1+b_1)^2+2a_2^2+2(a_2+b_2)^2)}}{(kd^3)^{2d^2}} B^{a_1+a_2-\frac{r+2}{r}(b_1+b_2)}$$

Thus,

$$\frac{\tilde{E}[x^{a_1+a_2}y^{b_1+b_2}]}{\sqrt{\tilde{E}[x^{2a_1}y^{2b_1}]\tilde{E}[x^{2a_2}y^{2b_2}]}} \leq (kd^3)^{-((a_1-a_2)^2+(a_1+b_1-a_2-b_2)^2)}$$

Since we either have that $a_1 \neq a_2$ or $a_1 + b_1 \neq a_2 + b_2$, this is at most $\frac{1}{kd^3} \leq \frac{1}{2d^2}$.

If $b_1 + b_2 > a_1 + a_2$ then we can use a symmetrical argument. If $a_1 + a_2 = b_1 + b_2$ and $a_1 > b_1$ then

$$\sqrt{\tilde{E}[x^{2a_1}y^{2b_1}]\tilde{E}[x^{2a_2}y^{2b_2}]} = \frac{(kd^3)^{(2a_1^2+2(a_1+b_1)^2+2b_2^2+2(a_2+b_2)^2)}}{(kd^3)^{2d^2}} B^{a_1+b_2-\frac{r+2}{r}(b_1+a_2)}$$

Since $a_1 + b_2 \geq b_1 + a_2 + 2$ we can't have $b_1 = a_2 = r$, $a_1 + b_2 - \frac{r+2}{r}(b_1 + a_2) \geq \frac{1}{r}$. Since $B \geq (kd^3)^{(3d^3)}$,

$$\frac{(kd^3)^{(2a_1^2+2(a_1+b_1)^2+2b_2^2+2(a_2+b_2)^2)}}{(kd^3)^{2d^2}} B^{a_1+b_2-\frac{r+2}{r}(b_1+a_2)} \geq \frac{(kd^3)^{\frac{1}{r}(3d^3)}}{(kd^3)^{2d^2}} \geq (kd^3)^{d^2}$$

Thus,

$$\frac{\tilde{E}[x^{a_1+a_2}y^{b_1+b_2}]}{\sqrt{\tilde{E}[x^{2a_1}y^{2b_1}]\tilde{E}[x^{2a_2}y^{2b_2}]}} \leq \frac{k^d}{(kd^3)^{d^2}} \leq \frac{1}{2d^2}$$

□

Lemma 6.9. For all a, b such that $a \neq b$, $a \leq r$, $b \leq r$, and $a + b \leq d$, $\tilde{E}[x^{a+b}y^{a+b}] \leq \frac{1}{4^{|a-b|}} \sqrt{\tilde{E}[x^{2a}y^{2a}]\tilde{E}[x^{2b}y^{2b}]}$

Proof. We have that $\sqrt{\tilde{E}[x^{2a}y^{2a}]\tilde{E}[x^{2b}y^{2b}]} \geq \frac{4^{2(a^2+b^2)}}{4^{(r^2)}} k^{a+b}$ while $\tilde{E}[x^{a+b}y^{a+b}] = \frac{4^{(a+b)^2}}{4^{(r^2)}} k^{a+b}$.

Thus,

$$\frac{\tilde{E}[x^{a+b}y^{a+b}]}{\sqrt{\tilde{E}[x^{2a}y^{2a}]\tilde{E}[x^{2b}y^{2b}]}} \leq \frac{1}{4^{(a-b)^2}} \leq \frac{1}{4^{|a-b|}}$$

□

Combining all of these results, it can be shown that the moment matrix M corresponding to \tilde{E} is PSD. We now bound $\sum_{I:|I|\leq d} (\tilde{E}[h_I])^2$. By a large margin, the dominant terms will come from the leading coefficients of the degree d Hermite polynomials for x and y . The leading coefficient of the degree d Hermite polynomial is $\frac{1}{\sqrt{d!}}$ so we have that

$$\sum_{I:|I|\leq d} (\tilde{E}[h_I])^2 \leq \frac{2}{\sqrt{d!}} B^{\frac{d}{2}} \leq (kd^3)^{2d^4}$$

By Lemma 2.12, for all g of degree at most d such that g is a sum of squares,

$$\|f - g\|^2 \geq \frac{(\tilde{E}[f])^2}{\sum_{I:|I|\leq d} (\tilde{E}[h_I])^2} = \frac{(c + (r+1)(1-k^r))^2}{(kd^3)^{2d^4}}$$

Taking $k = \sqrt[r]{2+c}$, the right hand side is at least $\frac{1}{(d^3 \sqrt[r]{2+c})^{2d^4}}$ and this completes the proof

□

Remark 6.10. *In fact, we could have taken any polynomial f on a constant number of variables which is non-negative but not a sum of squares and this covers the case of polynomials with degree $0 \pmod 4$ (the above construction only gives us polynomials of degree $2 \pmod 4$). However, it would be preferable to have an example which really depends on all its variables. We leave this as a question for future work. Also, the constant is a rapidly decaying function of d and it would be very interesting to obtain a more reasonable constant.*

7 Future Work

In this paper, we have shown that property testing whether a polynomial is a sum of squares using random samples and a tester which always accepts YES instances is hard; we need $n^{\Omega(d)}$ samples, which is not much less than we would need to completely determine the polynomial. That said, this work raises a number of questions, including but not limited to the following:

1. What can be shown for adaptive sampling and/or testers which only need to accept YES instances with high probability?
2. What is the threshold at which polynomial interpolation is likely to result in a polynomial with high norm? In other words, what is threshold at which M stops being close to the identity?
3. Can we obtain almost tight bounds on $\|g^c\|$ for $c \geq 2$ for polynomial interpolation on random points?
4. If a degree d polynomial f is far from being a degree d non-negative polynomial, must it be negative on a constant proportion of inputs?
5. Can we find a degree d polynomial which is non-negative, far from being a sum of squares, and is far from being a junta (even after a change in coordinates)? Can we property test whether there is some basis in which a polynomial f is a junta?
6. For a given d , is there a constant $\epsilon \in (0, 1)$ where there is a more efficient way to property test whether a polynomial f of norm 1 is a sum of squares or is ϵ -far from being a sum of squares?

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A Example: 4-XOR polynomial

In this section, we briefly discuss an attempt at creating a polynomial based on 4-XOR which is far from being non-negative yet passes the property test and why it does not quite work. The polynomial is constructed as follows.

1. Randomly choose $n^{2-\delta}$ equations of the form $x_I = b_I$ where I consists of 4 distinct elements of $[1, n]$ and $b_I \in \{-1, +1\}$.
2. Take the polynomial $p = \sum_I -b_I x_I$

As shown by Grigoriev [10], later rediscovered by Schoenebeck [18], and explained in Boaz Barak's lecture notes [4], we can construct pseudo-expectation values for a constant $d \geq 4$ as follows:

1. Start with the equations $x_I = b_I$ for every I which was chosen.
2. As long as there are sets I, J of size at most d such that $|I \Delta J| \leq d$ and we have not yet set $b_{I \Delta J} = b_I b_J$, set $b_{I \Delta J} = b_I b_J$. If this gives a contradiction because we already set $b_{I \Delta J} = -b_I b_J$, halt and fail. However, with high probability this will not happen.
3. Once we are done, we define \tilde{E} as follows.
 - (a) For all sets I of size at most d , take $\tilde{E}[x_I] = b_I$ if b_I was set and take $\tilde{E}[x_I] = 0$ otherwise.
 - (b) For all sets I of size between $d + 1$ and $2d$, we set $\tilde{E}[x_I] = 0$.
 - (c) For all multisets I such that $|I| \leq 2d$ and I contains some x_i with multiplicity 2, we take $\tilde{E}[x_I] = \tilde{E}[x_{I \setminus \{x_i, x_i\}}]$

Observe that $\tilde{E}[p] = -n^{2-\delta}$, so p is very far from being a sum of squares. In fact, using Lemma 2.12, it can be shown that with high probability, p is $(1 - o(1))\|p\|$ -far from being a sum of squares. If we could add a sum of squares polynomial g to p so that $\|g + p\|$ is $O(\|p\|)$ and with high probability polynomially many random samples of $g + p$ will all have nonnegative values, then this would give another example of a polynomial which passes our property tester with high probability yet is far from being a sum of squares. However, this may not be possible. As a special case, if we try taking $g = C$ for a constant C then we would need $C \gg \|p\|$ in order to make it so that with high probability, polynomially many random samples of $C + p$ all have nonnegative values.